# 14

# *Shape of Spectral Lines*

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If we take the classical picture of the atom as the definitive view of the formation of spectral lines, we would conclude that these lines should be delta functions of frequency and appear as infinitely sharp black lines on the stellar spectra. However, many processes tend to broaden these lines so that the lines develop a characteristic shape or profile. Some of these effects originate in the quantum mechanical description of the atom itself. Others result from perturbations introduced by the neighboring particles in the gas. Still others are generated by the motions of the atoms giving rise to the line. These motions consist of the random thermal motion of the atoms themselves which are superimposed on whatever large scale motions may be present. The macroscopic motions may be highly ordered, as in the case of stellar rotation, or show a high degree of randomness such as is characteristic of turbulent flow.

In practice, all these effects are present and give the line its characteristic shape. The correct representation of these effects allows for the calculation of the observed line profile and in the process reveals a great deal about the conditions in the star that give rise to the spectrum. Of course the photons that give rise to the absorption lines in the stellar spectrum have their origins at different locations in the atmosphere. So the conditions giving rise to a spectral line are really an average of a range of conditions. Thus, when we talk of the excitation temperature or the kinetic temperature appropriate for a specific spectral line, it must be clear that we are referring to some sort of average temperature appropriate for that portion of the atmosphere in which most of the line photons originate. For strong lines with optical depths much greater than the optical depth of the adjacent continuum, the physical depth of the line-forming region is quite small, and the approximation of the physical conditions by their average value is a good one. Unfortunately, for very strong lines, the optical depths can range to such large values that the line-forming region is located in the chromosphere, where most of the assumptions that we have made concerning the structure of the stellar atmosphere break down. A discussion of such lines will have to wait until we are ready to relax the condition of LTE.

In describing the shape or profile of a spectral line, we introduce the notion of the atomic line absorption coefficient. This is a probability density function that describes the probability that a given atom in a particular state of ionization and excitation will absorb a photon of frequency $\nu$ in the interval between $\nu$ and $\nu + d\nu$. We then assume that an ensemble of atoms will follow the probability distribution function of the single atom and produce the line. In order to make the connection between the behavior of a single atom and that of a collection of atoms, we shall make use of the Einstein coefficients that were introduced in Section 11.3.

## 14.1 Relation between the Einstein, Mass Absorption, and Atomic Absorption Coefficients

Since the Einstein coefficient $B_{ik}$ is basically the probability that an atom will make a transition from the ith state to the kth state in a given time interval, the relationship to the mass absorption coefficient can be found by relating all upward transitions to the total absorption of photons that must take place. From the definition of the Einstein coefficient of absorption, the total number of transitions that take place per unit time is

$$N_{i6k} = n_i B_{ik} I_\nu dt$$

$$(14.1.1)$$

where $n_i$ is the number density of atoms in the ith state. Since the number of photons available for absorption at a particular frequency is $(I_\nu/h\nu)d\nu$, the total number of upward transitions is also

$$N_{i \to k} = 4\pi n_i \int \kappa_\nu \rho \, \frac{I_\nu}{h\nu} \, d\nu \, dt$$

(14.1.2)

If we assume that the radiation field seen by the atom is relatively independent of frequency throughout the spectral line, then the integral of the mass absorption coefficient over the line is

$$\int \kappa_\nu \, d\nu = \frac{1}{4\pi} \frac{h\nu_0}{\rho} B_{ik}$$

(14.1.3)

where $\nu_0$ is the frequency of the center of the line.

For the remainder of this chapter, we will be concerned with the determination of the frequency dependence of the line absorption coefficient. Thus, we will be calculating the absorption coefficient of a single atom at various frequencies. We will call this absorption coefficient the *atomic line absorption coefficient*, which is related to the mass absorption coefficient by

$$S_\nu = \frac{S_\omega}{2\pi} = \kappa_\nu \rho / n_i$$

(14.1.4)

Note that we will occasionally use the circular frequency $\omega$ instead of the frequency $\nu$, where $\omega = 2\pi\nu$.

## 14.2 Natural or Radiation Broadening

Of all the physical processes that can contribute to the frequency dependence of the atomic line absorption coefficient, some are intrinsic to the atom itself. Since the atom must emit or absorb a photon in a finite time, that photon cannot be represented by an infinite sine wave. If the photon wave train is of finite length, it must be represented by waves of frequencies other than the fundamental frequency of the line center $\nu_0$. This means that any photon can be viewed in terms of a "packet" of frequencies ranging around the fundamental frequency. So the photon will consist of energy occupying a range of wavelengths about the line center. The extent of this range will depend on the length of the photon wave train. The longer the wave-train, the narrower will be the range of frequencies or wavelengths required to represent it.

Since the length of the wave train will be proportional to the time required to emit or absorb it, the characteristic width of the range will be proportional to the transition probability (i.e., the inverse of the transition time) of the atomic transition. This will be a property of the atom alone and is known as the *natural width* of the transition. It is always present and cannot be removed. Its existence depends only on the finite length of the wave train and so is not just the result of the quantum nature of the physical world. Indeed, there are two effects to estimate: the classical effect relying on the finite nature of the wave train, and the quantum mechanical effect that

can be obtained for a specific atom's propensity to emit photons. The former will be independent of the type of atom, while the latter will yield a larger broadening that depends specifically on the type of atom and its specific state.

### a      Classical Radiation Damping

The classical approach to the problem of absorption relies on a picture of the atom in which the electron is seen to oscillate in response to the electric field of the passing photon. There is a strong analogy here between the behavior of the electrons in the atom and the free electrons in an antenna. The energy of the passing wave is converted to oscillatory motion of the electron(s), which in the antenna produce a current that is subsequently amplified to signal the presence of the photon. It then makes sense to use classical electromagnetic theory to estimate this effect for the single optical electron of an atom. The oscillation of this electron can then be viewed as a classical oscillating dipole.

Since an oscillating electron represents a continuously accelerating charge, the electron will radiate or absorb energy. In the classical picture, the processes of emission and absorption are interchangeable. The emission simply requires the presence of a driving force, which is the ultimate source of the energy that is emitted, while the energy source for the absorption processes is the passing photon itself. If we let $\overline{W}$ represent the energy gained or lost over one cycle of the oscillating dipole, then any good book on classical electromagnetism (i.e., W. Panofsky and M. Phillips[1] or J. Slater and N. Frank[2]) will show that

$$\frac{d\overline{W}}{dt} = -\frac{2e^2}{3c^3}\overline{\left(\frac{d^2x}{dt^2}\right)^2}$$

(14.2.1)

where $d^2x/dt^2$ is the acceleration of the oscillating charge. Now if we assume that the oscillator is freely oscillating, then the instantaneous acceleration is simply

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$

(14.2.2)

This is a good assumption as long as the energy is to be absorbed on a time scale that is long compared to the period of oscillation. Since the driving frequency of the oscillator is that of the line center, this is equivalent to saying that the spread or range of absorbed frequencies is small compared to the frequency of the line center.

Equation (14.2.2) can be used to replace the mean square acceleration of equation (14.2.1) to get

$$\frac{d\overline{W}}{dt} = -\frac{2e^2}{3c^3}\omega_0^4\overline{x^2}$$

(14.2.3)

The mean position of the oscillator $\overline{x^2}$ can, in turn, be replaced with the mean total

energy of the oscillator from

$$\bar{W} = \langle T \rangle + \langle \Phi \rangle = m\omega_0^2 \overline{x^2}$$

(14.2.4)

so that the differential equation for the absorption or emission of radiation from a classical oscillating dipole is

$$\frac{d\bar{W}}{dt} = -\frac{2e^2\omega_0^2\bar{W}}{3m_e c^3} = -\gamma\bar{W}$$

(14.2.5)

The quantity $\gamma$ is known as the *classical damping constant* and is

$$\gamma = \frac{2e^2\omega_0^2}{3m_e c^3}$$

(14.2.6)

The solution of equation (14.2.5) shows that the absorption of the energy of the passing photon will be

$$I = I_0 e^{-\gamma t}$$

(14.2.7)

where $I_0$ is the presumably sinusoidally varying energy field of the passing photon. The result is that energy of the absorbed or emitted photon resembles a damped sine wave (see Figure 14.1).

But, we are interested in the behavior of the absorption with wavelength or frequency, for that is what yields the line profile. Since we are interested in the behavior of an uncorrelated collection of atoms, their combined effect will be proportional to the combined effect of the squares of the electric fields of their emitted photons. Thus, we must calculate the Fourier transform of the time-dependent behavior of the electric field of the photon so that

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E(t)e^{i\omega t}\, dt$$

(14.2.8)

If we assume that the photon encounters the atom at $t = 0$ so that $E(t) = 0$ for $t < 0$, and that it has a sinusoidal behavior $E(t) = E_0 e^{-i\omega_0 t}$ for $t \geq 0$, then the frequency dependence of the photon's electric field will be

$$E(\omega) = \frac{E_0}{\sqrt{2\pi}} \int_0^{\infty} e^{-[t\gamma/2 + i(\omega - \omega_0)t]}\, dt$$

(14.2.9)

Thus the power spectrum of the energy absorbed or emitted by this classical oscillator will be

$$I_\omega \propto E_\omega^2 = (\text{const}) \left[ (\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2 \right]^{-1}$$
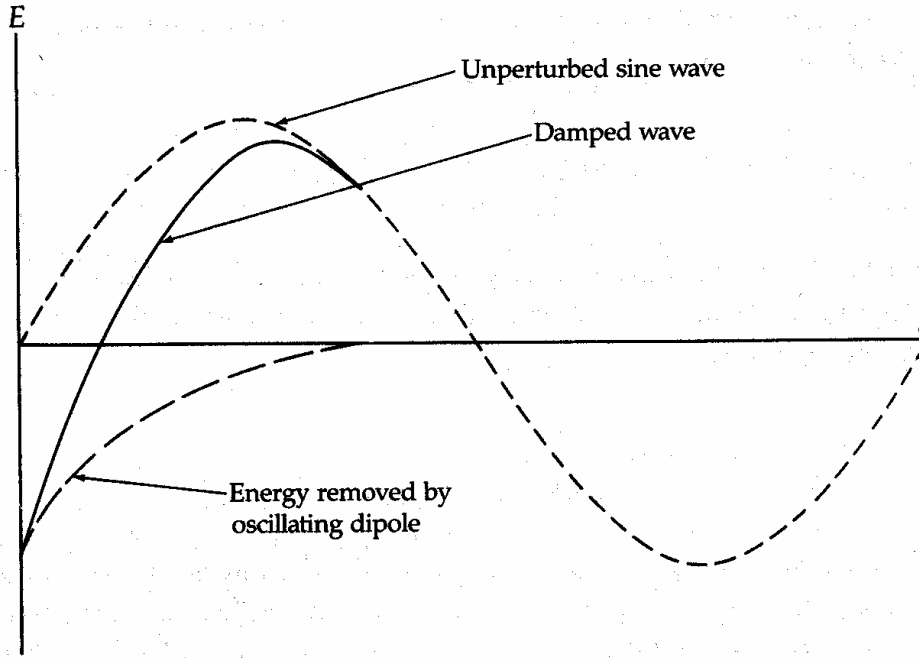
$$(14.2.10)$$



**Figure 14.1** is a schematic representation of the effect of radiation damping on the wave train of an emitted (absorbed, if t is replaced with -t) photon. The pure sine wave is assumed to represent the photon without interaction, while the exponential dotted line depicts the effects of radiation damping by the classical oscillator. The solid curve is the combined result in the time domain.

It is customary to normalize this power spectrum so that the integral over all frequencies is unity so that

$$I_\omega = \frac{\gamma/(2\pi)}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

$$(14.2.11)$$

This normalized power spectrum occurs frequently and is known as a *damping profile* or a *Lorentz profile*. Since the atomic absorption coefficient will be proportional to the energy absorbed,

$$S_\omega = \frac{\pi e^2}{m_e c} \frac{\gamma}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

$$(14.2.12)$$

Here the constant of proportionality can be derived from dispersion theory[3]. A plot of $S_\omega$ shows a hump-shaped curve with very large "wings" characteristic of a damping

profile (see Figure 14.2). At some point in the profile, the absorption coefficient drops to one-half of its peak value. If we denote the full width at this half-power point by $\Delta\lambda_c$, then

$$\Delta\lambda_c = \frac{2\pi c\gamma}{\omega_0^2} = \frac{4\pi e^2}{3m_e c^2} \approx 1.18 \times 10^{-4} \text{ Å}$$

(14.2.13)

This is known as the *classical damping* width of a spectral line and is independent of the atom or line. It is also very much smaller than the narrowest lines seen in the laboratory, and to see why, we must turn to a quantum mechanical representation of radiation damping.
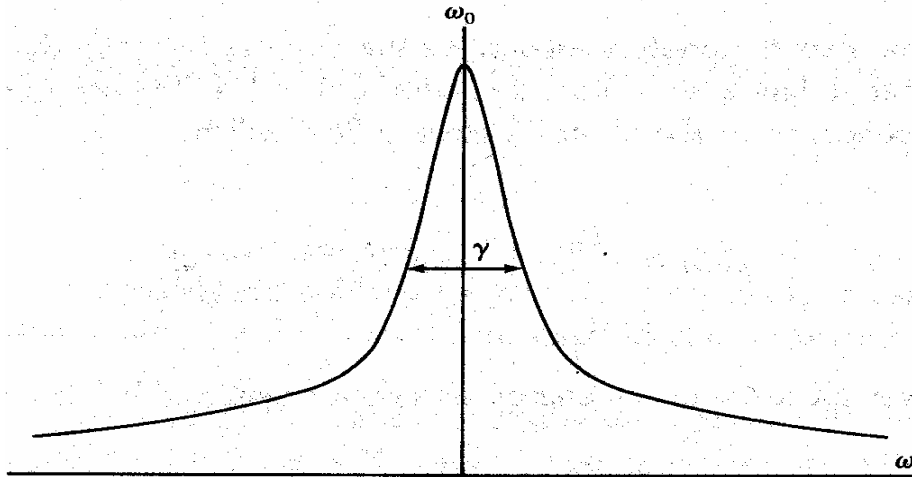


**Figure 14.2** shows the variation of the classical damping coefficient with wavelength. The damping coefficient drops to half of its peak value for wavelength shifts equal to $\Delta\lambda_c/2$ on either side of the central wavelength. The overall shape is known as the Lorentz profile.

**b        Quantum Mechanical Description of Radiation Damping**

The quantum mechanical view of the emission or absorption of a photon is rather different from the classical view since it is intimately connected with the nature of the atom in question. The basic approach involves the Heisenberg uncertainty principle as the basis of the broadening. If we consider an atom to be in a certain state, then the length of time that it can remain in that state is related to the uncertainty of the energy of that state by

$$\Delta E \, \Delta t \simeq \hbar$$

(14.2.14)

If there are a large number of states to which the atom can make a transition, then the probability of it doing so is great, Δt is small, and the uncertainty of the energy level is large. A large uncertainty in the energy of a specific state means that a wide range of frequencies can be involved in the transition into or out of that state. Thus any line resulting from such a transition will be unusually broad. Thus any strong line resulting from frequent transitions will also be quite broad.

This view of absorption and emission was quantified by Victor Weisskopf and Eugene Wigner[4,5] in 1930. They noted that the probability of finding an atom with a wave function $\Psi_j$ in an excited state $j$ after a transition from a state $i$ is

$$P_j(t) = \Psi_j^* \Psi_j e^{-\Gamma t}$$

(14.2.15)

where $\Gamma$ is the Einstein coefficient of spontaneous emission $A_{ji}$. The exponential behavior of $P_j(t)$ ensures that the power spectrum of emission will have the same form as the classical result, namely,

$$I(\omega) = \frac{\Gamma/(2\pi)}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

(14.2.16)

If the transition takes place between two excited levels, which can be labeled $u$ and $l$, the broadening of which can be characterized by transitions from those levels, then the value of gamma for each level will have the form

$$\Gamma_u = \sum_{i < u} A_{ui} \qquad \Gamma_l = \sum_{i < l} A_{li}$$

(14.2.17)

The power spectrum of the transition between them will then have the form of equation (14.2.16), but with the value of gamma determined by the width of the two levels so that

$$\Gamma = \Gamma_l + \Gamma_u$$

(14.2.18)

### c      Ladenburg f-value

Since the power spectrum from the quantum mechanical view of absorption has the same form as that of the classical oscillator, it is common to write the form of the atomic absorption coefficient as similar to equation (14.2.12) so that

$$S_\omega = \frac{\pi e^2}{m_e c} f_{ik} \frac{\Gamma_{ik}}{(\omega - \omega_0)^2 + (\Gamma_{ik}/2)^2}$$

(14.2.19)

The quantity $f_{ik}$ is then the equivalent number of classical oscillators that the transition from $i \rightarrow k$ can be viewed as representing. If you like, it is the number that brings the quantum mechanical calculation into line with the classical representation

of radiation damping. If the energy levels are broad, then the transition is much more likely to occur than one would expect from classical theory, the absorption coefficient will be correspondingly larger, and $f_{ik} > 1$. The quantity $f_{ik}$ is known as the *Ladenburg f value* or the *oscillator strength*. However, the line profile will continue to have the characteristic Lorentzian shape that we found for the classical oscillator.

Since the $f$ value characterizes the entire transition, we expect it to be related to other parameters that specify the transition. Thus, the $f$ value and the Einstein coefficient of absorption are not independent quantities. We may quantify this relation by integrating equation (14.2.19) over all frequencies and by using equation (14.1.4), substituting into equation (14.1.3) to get

$$\frac{\pi e^2 f_{ik}}{m_e c} \frac{1}{2\pi} \int_0^\infty \frac{\Gamma_{ik}\, d\omega}{(\omega - \omega_0)^2 + (\Gamma_{ik}/2)^2} = \frac{e^2 f_{ik}}{m_e c} \int_{-\omega_0}^\infty \frac{a\, dx}{a^2 + x^2}$$

$$= B_{ik} \frac{h\nu_0}{4\pi}$$

(14.2.20)

where $a = \Gamma_{ik}/2$, and $\nu_0$ is the frequency of the line center. If we make the assumption that the line frequency width is small compared to the line frequency, then $\Gamma_{ik}/\omega_0 \ll 1$ and equation (14.2.20) becomes

$$f_{ik} = \frac{m_e c}{\pi e^2} \frac{h\nu_0 B_{ik}}{4\pi}$$

(14.2.21)

Thus the classical atom can be viewed as radiating or absorbing a damped sine wave whose Fourier transform contains many frequencies in the neighborhood of the line center. These frequencies are arranged in a symmetrical pattern known as a *Lorentz* or *damping profile* characterized by a specific width. The quantum mechanical view changes very little of this except that the transition can be viewed as being made up of a number of classical oscillators determined by the Einstein coefficient of the transition. In addition, the classical damping constant is replaced by a damping constant that depends on all possible transitions in and out of the levels involved in the transition of interest. The term that describes this form of broadening is *radiation damping* and it is derived from the damped form of the absorbed or emitted photon wave train, as is evident from the classical description.

The broadening of spectral lines by this process is independent of the environment of the atom and is a result primarily of the probabilistic behavior of the atom itself. In cases where external forms of broadening are small or absent, radiation damping may be the dominant form of broadening that effectively determines the shape of the spectral line. When this is the case, little about the nature of the environment can be learned from the line shape. However, for normal stellar atmospheres and most lines, perturbations caused by the surrounding medium cause changes in the energy levels that far outweigh the natural broadening of the uncertainty principle. We now consider these forms.

## 14.3   Doppler Broadening of Spectral Lines

The atoms that make up the gas of the stellar atmosphere are constantly in motion, and this motion shifts the wavelengths, seen by an observer, at which the atoms can absorb radiation. This motion may be only the thermal motion of the gas, or it may include the larger-scale motions of turbulence or rotation. Whatever the combination, the shifting of the rest wavelengths by varying amounts for different populations of atoms will usually result in the observed line's being broadened by an amount significantly greater than the natural width determined by atomic properties.

The shifting of the rest wavelength caused by the motion of the atoms not only produces a change as seen by the observer, but also may expose the atom to a somewhat different radiation field. This will be true if the motion is locally random so that the motion of each atom is uncorrelated with that of its neighbors. However, should the motions be large-scale, then entire collections of atoms will have their rest wavelengths shifted by the same amount with respect to the observer and the star. If these collections of atoms constitute an optically thick ensemble, then the radiation field of the ensemble will be shifted along with the rest wavelength. To atoms within such a "cloud" there will be no effect of the motion on the atoms themselves. It will be as if a "mini-atmosphere" was moving, and no additional photons will be absorbed as a result of the motion. Such motions will not affect the equivalent widths of lines but may change the profiles considerably.

Contrast this with the situation resulting from an atom whose motion is uncorrelated with that of its neighbors. Imagine a line with an arbitrarily sharp atomic absorption coefficient [that is, $S_\nu = \delta(\nu-\nu_0)$]. If there were no motion in the atmosphere, the lowest-lying atoms would absorb all the photons at frequency $\nu_0$, leaving none to be absorbed by the overlying atoms. Such a line is said to be saturated because the addition of absorbing material will make no change in the line profile or equivalent width. But, allow some motion, and the rest frequency of these atoms is changed slightly from $\nu_0$. Now these atoms will be capable of absorbing photons at the neighboring frequencies, and the line will appear wider and stronger. Its equivalent width will be increased simply as a result of the Doppler shifts experienced by some atoms. Thus, if the motion consists of collections of atoms that are optically thin, we can expect changes in the line strengths as well as in the profiles. However, if those collections of atoms are large enough to be optically thick, then no change in the equivalent width will occur in spite of marked changes in the line profile. We refer to the motions of the first case as *microscopic* motions so as to contrast them with the second case of *macroscopic* motion.

### a        Microscopic Doppler Broadening

Again, it is useful to make a further subdivision of the classes of microscopic motions based on the nature of those motions. In the case of thermal motions, we may make plausible assumptions regarding the velocity field of the atoms.

***Thermal Doppler Broadening***        The assumption of LTE from Section 9.1b stated that the particles that make up the gas obeyed Maxwell-Boltzmann statistics appropriate for the local values of temperature and density. For establishing the Saha-Boltzmann ionization and excitation formulas, it was really only necessary that the electrons dominating the collision spectrum exhibit a maxwellian energy spectrum. However, we will now insist that the ions also obey Maxwell-Boltzmann statistics so that we may specify the velocity field for the atoms. With this assumption, we may write

$$\frac{dN(v)}{N} = \frac{1}{\sqrt{\pi}} e^{-v^2/v_0^2} \frac{dv}{v_0}$$

(14.3.1)

where dN/N is just the fraction of particles having a speed lying between v and v + dv and so it is a probability density function of the particle energy distribution. It is properly normalized since the integrals of both sides of equation (14.3.1) are unity. The second moment of this energy distribution gives

$$\langle v^2 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v^2 e^{-(v/v_0)^2} \frac{dv}{v_0} = \tfrac{1}{2} v_0^2$$

(14.3.2)

which we may relate to the kinetic energy of the gas.

Now we wish to pick the speed used in equations (14.3.1), and (14.3.2) to be the radial or line of sight velocity. Since there is no preferred frame of reference for the random velocities of thermal motion, this choice is as good as any other. However, the mean square velocity $\langle v^2 \rangle$ calculated in equation (14.3.2) is then only averaged over line-of-sight or radial motions and thus represents only 1 degree of freedom for the particles of the gas. So the energy associated with that motion is equal to ½kT for a monatomic gas, and

$$v_0^2 = \frac{2kT}{m}$$

14.3.3)

With the aid of the first-order (classical) Doppler shift, we define the Doppler width of a line in terms of $v_0$ as

$$\frac{\Delta\lambda_d}{\lambda_0} = \frac{\Delta v_d}{v_0} = \frac{v_0}{c}$$

(14.3.4)

Using equation (14.3.4), we may rewrite the particle distribution function for velocity as one for the fraction of atoms capable of absorbing at a frequency shift $\Delta v$ (or wavelength shift $\Delta\lambda$).

$$\frac{dN(\Delta\lambda)}{N} = \frac{1}{\sqrt{\pi}} e^{-(\Delta\lambda/\Delta\lambda_d)^2} \frac{d\lambda}{\Delta\lambda_d} = \frac{1}{\sqrt{\pi}} e^{-(\Delta v/\Delta v_d)^2} \frac{dv}{\Delta v_d} = \frac{dN(\Delta v)}{N}$$

(14.3.5)

Since the atomic line absorption coefficient is basically the probability of an atom's absorbing a photon at a given frequency, that probability should be proportional to the number of atoms capable of absorbing at that frequency. Thus,

$$S_v\, dv = \frac{A}{\sqrt{\pi}} e^{-(\Delta v/\Delta v_d)^2} \frac{dv}{\Delta v_d}$$

(14.3.6)

where $A$ is simply a constant of proportionality. This constant can be related to the Einstein coefficient by equations (14.1.3), and (14.1.4), with the result that

$$S_v\, dv = \frac{hv_0 B_{ik}}{4\pi\sqrt{\pi}\,\Delta v_d} e^{-(\Delta v/\Delta v_d)^2}\, dv = \frac{\sqrt{\pi} e^2 f_{ik}}{m_e c\,\Delta v_d} e^{-(\Delta v/\Delta v_d)^2}\, dv$$

(14.3.7)

To get the result on the far right-hand side, we used the relationship between the $f$ value for a particular transition and the Einstein coefficient given by equation (14.2.21). This is the expression for the atomic line absorption coefficient for thermal Doppler broadening. It differs significantly from the Lorentz profile of radiation damping by exhibiting much stronger frequency dependence. A spectral line where both broadening mechanisms are present will possess a line core that is dominated by Doppler broadening while the far wings of the line will be dominated by the damping profile as the gaussian profile of the Doppler core rapidly goes to zero.

***Microturbulent Broadening***   In addition to the thermal velocity field, the atoms in the atmospheres of many stars experience motion due to turbulence. Unfortunately, the theory of turbulent flow is insufficiently developed to enable us to make specific predictions concerning the velocity distribution function of the turbulent elements. So, for simplicity, we assume that they also exhibit a maxwellian velocity distribution, but one having a characteristic velocity different from the thermal velocity. Thus, the form of the probability density distribution function for turbulent elements is the same as equation (14.3.1) except that the velocity is the radial velocity of the turbulent cell:

$$\frac{dN}{N} = \frac{1}{\sqrt{\pi}} e^{-(v/v_0)^2} \frac{dv}{v_0}$$

(14.3.8)

If there were no other processes to consider, the atomic absorption coefficient for a turbulently broadened line would have the same form as one that is thermally

broadened except for a minor change in the interpretation of the Doppler half-width. However, we are interested in the combined effects of thermal and turbulent broadening, and so we consider how this combination may be carried out.

Since equation (14.3.1) represents the fraction of particles with a thermal velocity within a particular range, we may write the probability that a given atom will have a thermal velocity lying between v and v + dv as

$$P_{\text{th}}(v) = \frac{N}{v_0\sqrt{\pi}} e^{-(v/v_0)^2}$$

(14.3.9)

The probability that this same atom will reside in a particular turbulent element having a turbulent velocity lying between $\upsilon$ and $\upsilon + d\upsilon$ can be obtained, in a similar manner, from equation (14.3.8) and is

$$p_{\text{turb}}(v) = \frac{1}{\upsilon_0\sqrt{\pi}} e^{-(\upsilon/\upsilon_0)^2}$$

(14.3.10)

However, the observer does not regard these velocities as being independent since she or he is interested only in those combinations of velocities that add to produce a particular radial velocity v which yields a Doppler-shifted line. So we must regard the thermal and turbulent velocities to be constrained by

$$\mathbf{v} = v + \upsilon$$

(14.3.11)

Now the joint probability that an atom will have a velocity v lying between v and v + dv resulting from specific thermal and turbulent velocities v and $\upsilon$, respectively, is given by the product of equations (14.3.9) and (14.3.10). But we are not interested in just the probability that a thermal velocity v *and* a turbulent velocity $\upsilon$ will yield an observed velocity v; rather we are interested in all combinations of v and $\upsilon$ that will yield v. Thus we must sum the product probability over all combinations of v and $\upsilon$ subject to the constraint given by equation (14.3.11). With this in mind, we can write the combined probability that a given atom will have combined thermal and turbulent velocities that yield a specific observed radial velocity as

$$p(\mathbf{v}) = \frac{dN}{dv} = \int_{-\infty}^{+\infty} P_{\text{th}}(v)p_{\text{turb}}(\mathbf{v} - v)\, dv$$

$$= \frac{N}{\pi v_0 \upsilon_0} \int_{-\infty}^{+\infty} e^{-(v/v_0)^2 - [(\mathbf{v} - v)/\upsilon_0]^2}\, dv$$

(14.3.12)

Since the velocities involved in equation (14.3.12) are radial velocities, they may take on both positive and negative values. Thus the range of integration must run from -∞ to +∞. After some algebra, equation (14.3.13) yields the fraction of atoms with a combined velocity v to be

$$\frac{dN}{N} = \frac{e^{-(v/v_0)^2}}{v_0\sqrt{\pi}}\,dv$$

(14.3.13)

where

$$v_0^2 = v_0^2 + v_0^2$$

(14.3.14)

The similarity of the form of equation (14.3.13) to that of equations (14.3.1), and (14.3.8) is no accident. The integral in equation (14.3.12) is known as a *convolution integral*. The combined probability of *p(a) and p(b)* involves taking the product *p(a) × p(b)*. If, in addition, one has a constraint *q(c) = q(a,b),* then he must consider all combinations of *a* and *b* that yield *c* and sum over them. That is, one wants the probability of $(a_1,b_1)$ *or* $(a_2,b_2)$ etc. that yields *c*. Combining probabilities of *A* or *B* involves summing those probabilities. So, in general, if one wishes to find the combined probability of two processes subject to an additional constraint, one "convolves" the two probabilities. It is a general property of convolution integrals where the probability distributions have the same form that the resultant probability will also have the same form with a variance that is just the sum of the variances of the two initial probability distribution functions. Thus the convolution of any two Gaussian distribution functions will itself be a Gaussian distribution function having a variance that is just the sum of the two initial variances. This explains the form of equation (14.3.14). As a result, we may immediately write the atomic absorption coefficient for the combined effects of thermal and turbulent Doppler broadening as

$$S_v = \frac{\sqrt{\pi}e^2 f_{ik}}{m_e c\,\Delta v_d}\,e^{-(\Delta v/\Delta v_d)^2}$$

(14.3.15)

where

$$\Delta v_d = \frac{v_0 v_0}{c}$$

(14.3.16)

and

$$v_0^2 = \frac{2kT}{m} + v_0^2$$

(14.3.17)

It is now clear why we assumed the turbulent broadening to have a Maxwellian velocity distribution. If this were not the case, the convolution integral would be more complicated. If the turbulent velocity distribution function had the form

$$p_{\text{turb}}(v) = \Phi(v)$$

(14.3.18)

then the convolution integral with thermal broadening would become

$$p(\mathrm{v}) = N \int_{-\infty}^{+\infty} \frac{\Phi(x)}{\sqrt{\pi} v_0} e^{-(x-v)^2/v_0^2} dx$$

(14.3.19)

If the function $\Phi(x)$ is sufficiently simple, the integral may be expressed in terms of analytic functions. If not, then the integral must be evaluated numerically as part of the larger calculation for finding the line profile.

***Combination of Doppler Broadening and Radiation Damping*** Any spectral line will be subject to the effects of radiation damping or some other intrinsic broadening mechanism as well as the broadening introduced by Doppler motions. So to get a reasonably complete description of the atomic absorption coefficient, we have to convolve the Doppler profile with the classical damping profile given by equation (14.2.19). However, since the atomic absorption coefficient is expressed in terms of frequency, the constraint on the independent variables of velocity and frequency must contain the Doppler effect of that velocity on the observed frequency. Thus the frequency $v'$ at which the atom will absorb in terms of the rest frequency $v_0$ is

$$v' = v_0 + \frac{v_0 \mathrm{v}}{c}$$

(14.3.20)

For an atom moving with a line-of-sight velocity v, the atomic absorption for radiation damping is

$$S_v = \frac{\pi e^2 f_{ik} \Gamma_{ik}}{4\pi^2 m_e c} \frac{1}{(v_0 + v_0 \mathrm{v}/c - v)^2 + [\Gamma_{ik}/(4\pi)]^2}$$

(14.3.21)

This atomic absorption coefficient is essentially the probability that an atom having velocity v will absorb a photon at frequency v. To get the total absorption coefficient, we must multiply by the probability that the atom will have the velocity v [equation (14.3.13)] and sum all possible velocities that can result in an absorption at v. Thus,

$$S_v = \frac{\pi e^2 f_{ik} \Gamma_{ik}}{4\pi m_e c} \int_{-\infty}^{+\infty} \frac{1}{v_0 \sqrt{\pi}} \frac{e^{-(\mathrm{v}/v_0)^2} d\mathrm{v}}{(v_0 + v_0 \mathrm{v}/c - v)^2 + [\Gamma_{ik}/(4\pi)]^2}$$

(14.3.22)

The convolution integral represented by equation (14.3.22) is clearly not a simple one. When one is faced with a difficult integral, it is advisable to change variables so that the integrand is made up of dimensionless quantities. This fact will remove all the physical parameters to the front of the integral, clarifying their role in the result, and reduce the integral to a dimensionless weighting factor. This also

facilitates the numerical evaluation of the integral since the relative values of all the parameters of the integrand are clear. With this in mind we introduce the following traditional dimensionless variables:

$$u \equiv \frac{c(v - v_0)}{v_0 v_0}$$

$$y \equiv \frac{v}{v_0}$$

$$a \equiv \frac{c\Gamma_{ik}}{4\pi v_0 v_0} = \frac{\Gamma_{ik}}{4\pi \Delta v_d}$$

(14.3.23)

Substituting these into equation (14.3.22), we get

$$S_v(u) = \frac{e^2 f_{ik} a}{m_e v_0 v_0 \sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2}$$

(14.3.24)

It is common to absorb all the physical parameters on the right-hand side of equation (14.3.24) into a single constant that has the units of an absorption coefficient so that

$$S_0 \equiv \frac{\sqrt{\pi} e^2 f_{ik}}{m_e v_0 v_0}$$

(14.3.25)

The remaining dimensionless function can be written as

$$H(a, u) \equiv \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2}$$

(14.3.26)

This is known as the *Voigt function*, and it allows us to write the atomic absorption coefficient in the following simple way:

$$S_\omega(u) = S_0 H(a, u)$$

(14.3.27)

For small values of the damping constant ($a < 0.2$), the Voigt function is near unity at the line center (that is, $u = 0$) and falls off rapidly for increasing values of $|u|$. For values of $|u|$ near zero the Voigt function is dominated by the exponential that corresponds to the Doppler core of the line. However, at larger values of $|u|$, the denominator dominates the value of the integral. This corresponds to the damping wings of the line profile.

Considerable effort has gone into the evaluation of the Voigt function because it plays a central role in the calculation of the atomic line absorption coefficient. One of the earliest attempts involved expressing the Voigt function as

$$H(a, u) = \sum_{i=0}^{\infty} a^i H_i(u)$$

(14.3.28)

where the functions $H_i(u)$ are known as the *Harris functions*[6]. More commonly one

finds the alternative function

$$V(a, u) = \frac{H(a, u)}{\sqrt{\pi}}$$

(14.3.29)

whose integral over all *u* is unity. This function is known as the *normalized Voigt function*. Extensive tables of this function were calculated by D.Hummer[7] and a reasonably efficient computing scheme has been given by G.Finn and D.Mugglestone[8]. However, with the advent of fast computers emphasis has been put on finding a fast and accurate computational algorithm for the Voigt function. The best to date is that given by J.Humíek[9]. This has been expanded by McKenna[10] to include functions closely related to the Voigt function. All this effort has made it possible to obtain accurate values for the Voigt function with great speed, making the inclusion of this function in computer codes little more difficult than including trigonometric functions.

### b        Macroscopic Doppler Broadening

The fact that each atom was subject to all the broadening mechanisms described above caused most of the problems in calculating the atomic absorption coefficient through the introduction of a convolution integral. This approach assumed that each atom could "see" other atoms subject to the different velocity sources. However, if the turbulent elements were sufficiently large that they themselves were optically thick, then each element would optically behave independently of the others. The line profiles of each would be similar, but shifted relative to the others by an amount determined by the turbulent velocity of the element. Indeed, this would be the case if any motions involving optically thick sections of the atmosphere were present.

The proper approach to this problem involves finding the locally emitted specific intensity, convolving it with the velocity distribution function and integrating the result over the visible surface of the star to obtain the integrated flux. This flux can then be normalized to produce the traditional line profile. However, since the macroscopic motions can affect the structure of the atmosphere, the problem can become exceedingly difficult and solvable only with the aid of large computers. In spite of this, much can be learned about the qualitative behavior of these broadening mechanisms from considering some greatly simplified examples. We discuss just two, the first involves motions of large sections of the atmosphere in a presumably uncorrelated fashion, and the second involves the correlated motion of the entire star.

*Broadening by Macroturbulence*    It would be a mistake to assume that turbulent elements only come in sizes that are either optically thick or thin. However, to gain some insight into the degree to which turbulence can affect a line profile, we

divide the phenomena into these two cases. We have already discussed the effects that small turbulent elements have on the resulting atomic line absorption coefficient (i.e., microturbulence), and we have seen that they lead to an increase in value of that parameter for all frequencies. Such is not the case for macroturbulence. The motion of optically thick elements cannot change the value of the atomic line absorption coefficient because the environment of a particular atom concealed within the turbulent element is unaffected by the motion of that element. Thus, each element behaves as a separate "atmosphere", producing its own line profile, which contributes to the stellar profile by an amount proportional to the ratio of the visible area of the element to that of the apparent disk of the star. Thus, the combining (or convolution) of line profiles occurs not on the atomic level as with microturbulent Doppler broadening, but after the radiative transfer has been locally solved to yield a local line profile. This requires that we make assumptions that apply globally to the entire star in order to relate one turbulent element to another.

To demonstrate the nature of this effect, we consider a particularly simple situation where there is no limb-darkening in or out of the line. In addition, we assume that the local line profile is given by a Dirac delta function of frequency and that the macroturbulent motion is purely radial with a velocity $\forall v_m$. Under these conditions, zones of constant radial velocity will appear as concentric circles on the apparent disk (see Figure 14.3).

Since the intrinsic line profile is a delta function of frequency, the line profile originating at a ring of constant radial velocity located at an angle $\theta$ measured from the center of the disk will be Doppler shifted by an amount

$$\Delta v = v - v_0 = \pm \frac{v_m v_0 \mu}{c}$$

(14.3.30)

where, as usual,

$$\frac{F_c - F_v(\text{line})}{F_c} = 1 - r_v \propto 1 - \mu^2 = 1 - \left(\frac{\Delta v\, c}{v_0 v_m}\right)^2$$

(14.3.31)

The amount of energy removed from the total continuum flux by the local line absorption will simply be proportional to the area of the differential annulus located at the particular value of $\mu$ corresponding to $\Delta v$. Thus,

$$\frac{F_c - F_v(\text{line})}{F_c} = 1 - r_v \propto [1 - \mu^2]^{1/2} = 1 - \left(\frac{\Delta v\, c}{v_0 v_m}\right)^2$$

(14.3.32)

Therefore, the line profile would be given by

$$r_v = 1 - (1 - r_0)\left[1 - \left(\frac{\Delta v \, c}{v_m v_0}\right)^2\right]^{1/2} \qquad \Delta v \le \frac{v_0 v_m}{c}$$

(14.3.33)

This line profile is dish-shaped and is characteristic of this type of mass atmospheric motion. Since the equivalent width remains constant for macroscopic broadening, the central depth of the line will decrease for increasing $v_m$.
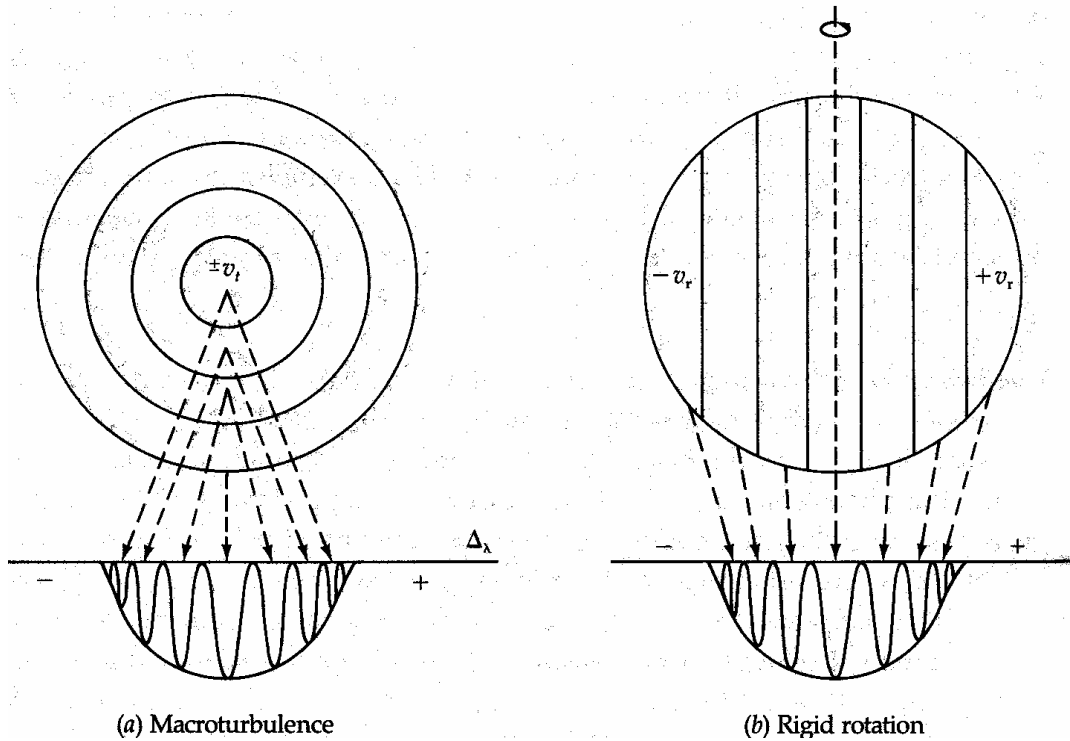


(a) Macroturbulence      (b) Rigid rotation

**Figure 14.3** schematically indicates the apparent disks of two idealized stars. Panel (*a*) depicts the lines of constant line-of-sight velocity for a macroturbulent stellar atmosphere where the turbulent motion is assumed to be along the stellar radius and of a fixed magnitude $v_m$. Panel (*b*) also indicates the lines of constant radial velocity for a spherical star that is spinning rigidly.

   Clearly a real situation replete with limb-darkening, a velocity dispersion of the turbulent elements, an anisotropic velocity field, along with a spectrum of sizes for the turbulent eddies, would make the problem significantly more difficult. A great deal of work has been done to treat the problem of turbulence in a more complete manner, but the results are neither simple to discuss nor easy to review. D.Mihalas[11] gives an introduction and excellent references to this problem.

   ***Broadening by Stellar Rotation***      As we saw in Chapter 7, rapid rotation of the entire star will lead to significant distortion of the star and a wide variation of

the parameters that define a stellar atmosphere over its surface. In such a situation, most of the assumptions we have made for the purpose of modeling the atmosphere no longer apply and recourse must be made to a more numerical approach (see G.Collins[12] and J.Cassinelli[13]). However, as with macroturbulence, some insight may be gained by considering the effects of rotation on the line profile of a slowly rotating star. Such a model is originally due to G.Shajn and O.Struve[13] and is now commonly referred to as the *Struve model.*

Consider a uniformly bright spherical star which is rotating as a solid body. Except for the rotation, this is essentially the same model as that used for the discussion of macroturbulence (see Figure 14.3). If we define $\theta$ and $\phi$, respectively, to be the polar and azimuthal angles of a spherical coordinate system with its polar axis aligned with the rotation axis of the star, then the velocity toward the observer's line of sight is

$$v_r = v_{eq} \sin \theta \sin \phi \sin i$$

(14.3.34)

where $v_{eq}$ is the equatorial velocity of the star and $i$ is the angle between the line-of-sight and the rotation axis, called the *inclination.* An inspection of Figure 14.3 and some geometry leads one to the conclusion that for spherical stars the product $\sin i$ $\sin \phi$ is constant on the stellar surface along any plane parallel to the meridian plane. Thus, any chord on the apparent disk that is parallel to the central meridian is a locus of constant radial velocity (see Figure 14.3). Any profile formed along this cord will be displaced in frequency by an amount

$$\Delta v = \frac{v_{eq} v_0}{c} (1 - \mu^2)^{1/2} \sin i = a(1 - \mu^2)^{1/2}$$

(14.3.35)

For a sphere of unit radius, the length of the chord is $2\mu$. If we make the same assumptions about the intrinsic line profile as were made for the case of macroturbulence (i.e., it can be locally represented by a delta function), then the amount of flux removed from the continuum intensity by any profile located on one of these chords will just be proportional to the length of the chord. Therefore,

$$\frac{F_c - F_v(\text{line})}{F_c} = 1 - r_v \propto 2\mu = 2\left(1 - \frac{\Delta v^2}{a^2}\right)^{1/2}$$

(14.3.36)

which leads to a profile of the form

$$r_v = 1 - (1 - r_0)\left(1 - \frac{\Delta v^2 c^2}{v_0^2 v_{eq}^2 \sin^2 i}\right)^{1/2} \qquad \Delta v \leq \frac{v_0 v_{eq} \sin i}{c}$$

(14.3.37)

Except for the replacement of the turbulent velocity by the equatorial velocity, the rotational profile has the same form as the profile for macroturbulence [equation

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(14.3.33)]. This points out a fundamental problem of Doppler broadening by mass motions. In general, it is not possible to infer the velocity field from the line profile alone. To be sure, the presence of limb-darkening would affect these two cases differently, as would the introduction of gravity darkening for the case of rotation. But the non-uniqueness remains for the general case, and any determination of the velocity field from the analysis of line profiles is strongly model-dependent and usually relies on some assumed symmetry.

Many of the simplifying assumptions of these models for macroturbulence and rotation can be removed for a modest increase in complexity. In the case of rotation, if the local line profile were not given by a delta function but had an intrinsic shape r'(*x*) where

$$x = \frac{\Delta v \, c}{v_0 v_{\text{eq}}}$$

(14.3.38)

then the observed line profile would be given by the convolution integral

$$1 - r(x) = \int_{-1}^{+1} [1 - r'(x - y)] Q(y) \, dy$$

(14.3.39)

Here Q(*y*) is known as the *rotational broadening function* which, if limb-darkening is included, is given by A.Unsöld[15] as

$$Q(y) = \frac{3}{3 + 2\beta} \left[ \frac{2}{\pi} (1 - y^2)^{1/2} + \frac{\beta}{2} (1 - y^2) \right]$$

(14.3.40)

The parameter β is the first-order limb-darkening coefficient. Consider the case for β = 0 and that the intrinsic line profile is a delta function. It is clear that equations (14.3.39) and (14.3.40) will yield equation (14.3.37) as long as the integral of Q(*y*) is normalized to unity. Integration of equation (14.3.40) will satisfy the skeptic that this is indeed the case. It is also clear that the general effects of rotation are not qualitatively very different from those implied by equation (14.3.37). While quantitative comparison with observation will clearly be affected by such things as the intrinsic line profile and limb-darkening, a truly useful comparison will have to go even further and include the effects of the variation of the atmospheric structure over the surface on the line profile.

While macroturbulence and rotation constitute the most important forms of macroscopic broadening, there are others. The presence of magnetic fields can split atomic lines through the Zeeman effect. In some instances, this can lead to anomalously broad spectral lines and subsequent errors in the abundances derived from these lines. In some instances, the broadening is sufficiently large to allow the estimation of the magnetic field itself. Fortunately, strong magnetic fields appear to be sufficiently rare among normal stellar atmospheres to allow us to ignore their

effects most of the time. However, we should be ever mindful of the possibility of their existence and of the effects that they can introduce in the shaping of spectral lines.

## 14.4    Collisional Broadening

To this point, we have described the broadening of spectral lines arising from intrinsic properties of the atom and the collective effects of the motion of these atoms. However, in all but the most extreme cases of macroscopic broadening, the most prominent source of broadening of spectral lines results from the interaction of the absorbing atom with neighboring particles of the gas. Since these particles are often charged (even the neutral atoms possess the potential field of an electric dipole), their potential will interact with that of the atomic nucleus which binds the orbiting electrons. This interaction will perturb the energy levels of the atom in a time-dependent fashion. The collective action of these perturbations on an ensemble of absorbing atoms is to broaden the spectral line. The details of this broadening depend on the nature of the atom and energy level being perturbed and the properties of the dominant perturber. All phenomena that fall into this general class of broadening mechanisms are usually gathered under the generic term *collisional broadening*. However, some authors refer to this concept or a subset of it as *pressure broadening*, on the grounds that there can be no collisions unless the gas has some pressure. The use of the different terms is usually not of fundamental importance, and the basic notion of what is behind them should always be kept in mind.

There is some confusion in the literature (and much more among students) regarding the terminology for describing these processes. Some of this results from a genuine confusion among the authors, but most derives from an unfortunate choice of terms to describe some aspects of the problem. You should keep clearly in mind what is being described during any discussion of this topic - the broadening of atomic energy levels resulting from the perturbations of neighboring particles. We adopt a variety of theoretical approaches to this problem, each of which has its own name. Care must be taken lest the name of the theoretical approach be confused with a qualitatively different type of broadening. We discuss perturbations introduced by different types of perturbers, each of which will produce a characteristic line profile for the absorbing gas. Each of these profiles has its own name so as to delineate the type of perturbation. However, they are all just perturbations of the energy levels. Each type will generally be discussed in a "vacuum", in that we assume that it is the only form of perturbation that exists, when in reality virtually all types of perturbations are present at all times and affect the energy levels. Fortunately, one of them usually does dominate the level broadening.

There are two main theoretical approaches to collisional broadening. One deals with the weak, but numerous, perturbations that cause small amounts of

broadening. The other is concerned with the large, but infrequent, perturbations that determine the shape of the wings of the line. It as somewhat unfortunate that the former theoretical approach is known as *impact phase-shift theory*, while the latter is called the *statistical* or *static broadening theory*. The word impact conjures up visions of violence, yet the theoretical approach labeled by this word is concerned only with the weakest and least violent of the interactions. Similarly, the term *static* implies calm, but this approach deals with the most violent perturbations. So be it. We try to justify this apparent anomaly during the specific discussions of these approaches. In addition, we clearly label the myriad terms as they are introduced so that those which are synonymous are clearly separated from those which have unique meanings.

To estimate the perturbation to the atom that changes the energy of the transition and thereby broadens the line, we must characterize the nature of the collision. The two theoretical approaches to collisional broadening differ in this description. Both approaches are largely classical in form so that whatever is true for absorption is also true for emission. So we often deal with the effects of a collision on a radiating atom with the full intention of applying the results to absorption.

### a    Impact Phase-Shift Theory

The approach of impact phase-shift theory assumes that the collision is of a very short duration compared to the span of time during which the atom is actually radiating (or absorbing) the photon. Thus,

$$t_{col} \ll t_{rad}$$

(14.4.1)

It is the short duration of the collision that is responsible for the name impact for the theoretical approach.

***Determination of the Atomic Line Absorption Coefficient***    Suppose that the atom radiates in an undisturbed manner between collisions with a frequency $\omega_0$. The electric field of the emitted photon will vary as

$$E(t) = E_0 e^{-i\omega_0 t} \qquad -\frac{T}{2} \leq t \leq \frac{T}{2}$$

(14.4.2)

where $T$ is the time between collisions. Further assume that the radiation of the photon does not continue before or after the collision, so that

$$E(t) = 0 \qquad t > \left|\frac{T}{2}\right|$$

(14.4.3)

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It is this interruption in the emission of the photon, or at least a complete and discontinuous change in the phase of the emitted photon that terminates the wave train and provides the motivation for the second half of the name for this approach. Since a sine wave of finite length must contain wave components of higher frequency introduced by the discontinuity of the wave train, the emitted photon will have more components than the fundamental frequency and thus the line will appear to be broadened. To find this distribution in frequencies, we must take the Fourier transform of the temporal description of the electric field of the photon. So

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E(t) \; e^{i\omega t} \, dt = \frac{E_0}{\sqrt{2\pi}} \int_{-T/2}^{+T/2} e^{i(\omega - \omega_0)t} \, dt$$

(14.4.4)

or

$$E(\omega) = + \frac{E_0}{\sqrt{2\pi}} \frac{2 \sin[(\omega - \omega_0)T/2]}{\omega - \omega_0}$$

(14.4.5)

Since the power spectrum of the emitted photon will depend on the square of the electric field,

$$I(\omega) = \frac{A^2 E_0^2}{2\pi} \frac{\sin^2[T(\omega - \omega_0)/2]}{[(\omega - \omega_0)/2]^2}$$

(14.4.6)

Here we have assumed that we will be dealing with emissions and absorptions that are totally uncorrelated, which for random collisions occurring in a sea of unrelated atoms is a perfectly reasonable assumption.

Now to determine the effects of multiple collisions (or numerous atoms), we must combine the effects of these collisions, which means that we must have some estimate of the time between them $T$. Let $P(t)$ be the probability that a collision has *not* occurred in a time t measured from the last collision. Now if the collisions are indeed random, the differential probability $dp$ that a collision will occur in a time interval $dt$ is

$$dp = \frac{dt}{T_0}$$

(14.4.7)

where $T_0$ is the average time between collisions. Thus, the differential change in the probability $P(t)$ is

$$d[1 - P(t)] = -dP(t) = P(t)\,dp = \frac{P(t)\,dt}{T_0}$$

(14.4.8)

or

$$P(t) = Be^{-t/T_0}$$

(14.4.9)

Since a collision must occur at some time, we can determine the constant in equation (14.4.9) by normalizing that expression to unity and integrating over all time. Thus we see that the collision frequency distribution is a Poisson distribution of the form

$$P(t) = \frac{1}{T_0} e^{-t/T_0}$$

(14.4.10)

and the constant of proportionality in equation (14.4.6) is $1/T_0$.

To obtain the total energy distribution or power spectrum resulting from a multitude of collisions, we must sum the power spectra of the individual collisions multiplied by the probability of their occurrence. Thus,

$$\begin{aligned} I_t(\omega) &= \frac{A^2 E_0^2}{2\pi T_0} \int_0^\infty \frac{\sin^2 [T(\omega - \omega_0)/2]}{[(\omega - \omega_0)/2]^2} e^{-T/T_0} dT \\ &= \frac{A^2 E_0^2}{\pi T_0} \frac{1}{(\omega - \omega_0)^2 + (1/T_0)^2} \end{aligned}$$

(14.4.11)

We can use the same normalization process implied by equations (14.1.3), (14.1.4), and (14.2.21) to write the atomic absorption coefficient as

$$S_\nu(\text{col}) = \frac{\pi e^2 f_{ik}}{2\pi T_0 m_e c} \frac{1}{(\nu - \nu_0)^2 + (1/T_0)^2}$$

(14.4.12)

In going from equation (14.4.11) to (14.4.12), we have changed from circular frequency w to frequency n so that the appropriate factors of $2\pi$ must be introduced. The quantity $2/T_0$ is usually called the *collisional damping constant* so that

$$\Gamma_c = \frac{2}{T_0}$$

(14.4.13)

Since the form of equation (14.4.12) is identical to that of equation (14.3.21), we can immediately obtain the convolution of the collisional damping absorption coefficient with that for radiation damping by simply adding the respective damping constants:

$$\Gamma = \Gamma_{ik} + \Gamma_c$$

(14.4.14)

The combined absorption coefficient could then be convolved with that appropriate for microturbulent Doppler broadening producing a total line profile that is still a Voigt profile but with

$$a = \frac{c\Gamma}{4\pi v_0 \mathsf{v}_0}$$

(14.4.15)

where $\Gamma$ is the combined damping constant for radiation and collisional damping. However, before this result can be of any practical use, we must have an estimate of the collisional damping constant in terms of the state variables of the atmosphere.

### *Determination of the Collisional Damping Constant*

Determining the collisional damping constant is equivalent to determining the average time between collisions $T_0$. To do this, it is necessary to be quite specific about exactly what constitutes a collision. We follow a method originally due to Victor Weisskopf[16] and described by many authors[17]-[19]. Consider that the perturbation of an energy level $\Delta E$ caused by a passing perturber has the distance dependence

$$\Delta h v = \hbar \Delta \omega \simeq r^{-n}$$

(14.4.16)

which will produce a change in the frequency of the emitted photon of

$$\Delta \omega = \frac{2\pi C_n}{r^n}$$

(14.4.17)

The constant $C_n$ is known as the *interaction constant*, and it must be determined empirically from laboratory experiments involving the kinds of particles found in the collisions. Since all these collisions are mediated by the electromagnetic force, the typical interaction can be viewed as a "long range" one so that the short collisions [see equation (14.4.1)] refer to distant collisions where the colliding particle is located near its point of closest approach. This distance is commonly referred to as the *impact distance*, or *impact parameter*. Since the collision is short and the interaction weak, we can assume that the perturbing particle is largely unaffected by the encounter, and its path can be viewed as a straight line (see Figure 14.4). This assumption is usually referred to as the *classical path approximation* and it appears in one form or another in all theories of collisional broadening.

We wish to calculate the entire frequency shift caused by the collision because when the accumulated phase shift becomes large enough, it is reasonable to say that the wave train has been interrupted and a collision has occurred.
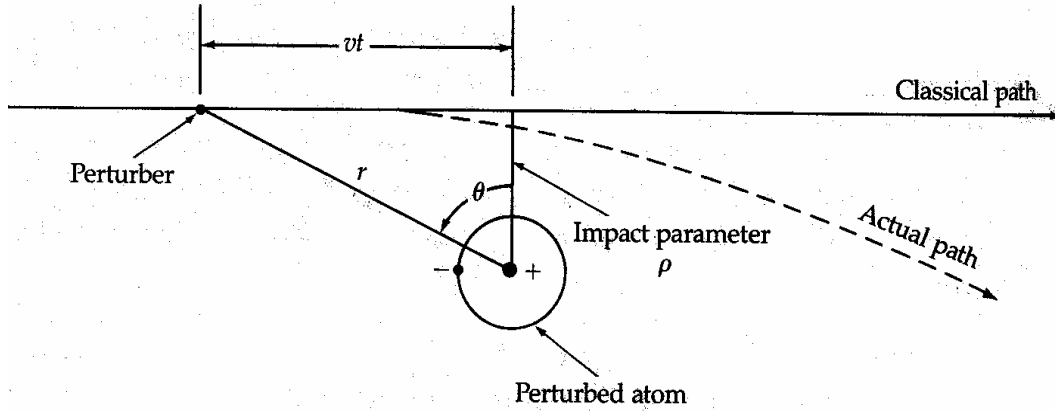
**Figure 14.4** shows the "classical path" taken by a perturbing particle under the Weisskopf approximation. The point of closest approach $\rho$ is called the impact parameter.

To estimate this total phase shift, it is necessary to describe the path taken by the particle, so we use of the classical path approximation. It is clear from Figure 14.4 that

$$r^2 = \rho^2 + (vt)^2$$

(14.4.18)

This enables the total phase shift $\eta$ caused by the encounter to be calculated from

$$\eta = \int_{-\infty}^{+\infty} \Delta\omega \, dt = 2\pi C_n \int_{-\infty}^{+\infty} \frac{dt}{[\rho^2 + (vt)^2]^{n/2}} = \frac{2\pi C_n a_n}{v\rho^{n-1}}$$

(14.4.19)

where

$$a_n = \frac{\sqrt{\pi}\,\Gamma[(n-1)/2]}{\Gamma(n/2)}$$

(14.4.20)

Here $\Gamma(x)$ is the gamma function, and it should not be confused with the symbol for the damping constant. Before using this for the determination of the average time between collisions, we must decide what constitutes an interruption in the wave train. Weisskopf took this value of $\eta$ to be 1 radian. The smaller the value for the phase shift, the larger the value of the impact parameter may be that will produce that phase shift. The value of the impact parameter $\rho_0$ that produces the minimum phase shift $\eta$ which constitutes an interruption in the wave train is known as the *Weisskopf radius*:

$$\rho_0 = \left(\frac{2\pi C_n a_n}{\eta_0 v}\right)^{1/(n-1)}$$

(14.4.21)

Since the Weisskopf radius defines the distance inside of which any encounters will produce a large enough phase shift to be considered a collision, it may be used to calculate a collision cross section $\sigma = \pi\rho_0^2$ and an average time between collisions $T_0$. The collision frequency is

$$\frac{1}{T_0} = \frac{\langle v \rangle_{\mathbf{rel}}}{l} = \pi\rho_0^2 N \langle v \rangle_{\mathbf{rel}} = \frac{\Gamma_c}{2}$$

(14.4.22)

where N is the number density of the perturbing particles, $l$ is the mean free path between collisions, and $\langle v \rangle_{rel}$ is the relative velocity between the perturber and the perturbed atom. That relative velocity is

$$\langle v \rangle_{\mathbf{rel}} = \left[ \frac{8kT}{\pi m_h} \left( \frac{1}{A_1} + \frac{1}{A_2} \right) \right]^{1/2}$$

(14.4.23)

where $A_i$ is the atomic weight of the constituents of the collision in units of the mass of the hydrogen atom. Thus, we can write the collisional damping constant as

$$\Gamma_c = (\mathbf{const}) \langle v \rangle_{\mathbf{rel}}^{1 - 2/(n-1)} \eta_0^{-2/(n-1)} N$$

(14.4.24)

where N is the number density of perturbers and

$$\mathbf{const} = (2\pi)^{(n+1)/(n-1)} (a_n C_n)^{2/(n-1)}$$

(14.4.25)

All that remains is to specify the power law that describes the perturbing force and the interaction constant $C_n$. Since the force that mediates the collision is electromagnetic, the exponent of the perturbing field is determined by the electric field of the perturber. A simple way of understanding this is to view the passage of the perturber as interposing a "screening" potential energy between an optical electron and the nucleus. The screening potential energy will depend on the locally interposed energy density of the perturber's electric field which is proportional to $E^2$ so that for a perturbing ion or electron, $n = 4$. This is called the *quadratic Stark effect* because it depends quadratically on the perturber's electric field. If the perturber is a neutral atom, it still possesses a dipole moment that produces a measurable field near the particle. However, this field varies as $r^{-3}$ so that the perturbing energy density varies as $r^{-6}$ and $n = 6$. Broadening of this type is called *van der Waal's broadening* and it will to play a role in relatively cool gases where there are few ions.

If the atomic energy level of interest is degenerate, the interposition of an external electric field will result in the removal of the degeneracy and a splitting of the energy level into a set of discrete energy levels. The amount of the splitting is proportional to the electric field. Since a time-dependent splitting is equivalent to a broadening brought about by a shift of the energy level itself, the broadening of degenerate levels will occur, but their broadening will be directly proportional to the

electric field of the perturber rather than to its square. Thus the broadening of a degenerate level by ions or electrons will produce $n = 2$. This form of broadening is known as the *linear Stark effect*. This form of broadening creates an interesting problem for the impact phase-shift theory since the integral for the minimum phase shift [equation (14.4.19)] will not converge for $n = 2$ and the theory is not applicable. Since the energy levels of hydrogen are degenerate, and the hydrogen lines are among the most prominent in stellar spectra, we are left with the somewhat embarrassing result that these lines can not be dealt with by the impact phase-shift theory and we have to resort to some other description of collisional broadening to obtain line profiles for hydrogen. The problem basically arises from the $1/r^2$ nature of the perturbing field and is not restricted to the theory of line broadening. Since the number of perturbers increases as $r^2$ while the perturbation from any one of them declines as $r^2$, the contribution to the total perturbation from particles at a given distance is independent of distance. Thus some cutoff of the distances to be considered must be invoked. This problem arises frequently in gravitation theory where there can be no screening of the potential field and the fundamental force is also long-range. In our case, the Heisenberg uncertainty principle sets a limit on the smallest perturbation that can matter and hence an upper limit on the volume of space to be considered.

The case of the broadening of degenerate levels by neutral atoms does present a situation that can be dealt with by the impact phase-shift theory. Here the electric field near the perturber varies as $r^{-3}$, so that the proper value of $n$ is $n = 3$. In the special case where the broadening is by atoms of the same species as the atom being perturbed a significant enhancement of the broadening occurs. Indeed, for astrophysical cases, the broadening of spectral lines arising from degenerate levels by neutral particles is of interest only when the broadening occurs from collisions with atoms of the same species. For that reason, this kind of broadening is known *as self-broadening*. These considerations are summarized in table 14.1.

An important improvement was made by Lindholm[20] and Foley[21] which included the effects of multiple collisions on the line. Although the multiple collisions are weak, they are frequent. The result of their work is that the secondary collisions introduce a slight shift in the line center of the atomic absorption coefficient so that

$$S_\omega(\text{col}) = \frac{2\pi e^2 f_{ik}}{mc} \frac{\Gamma_c/2}{(\omega - \omega_0 - \beta)^2 + (\Gamma_c/2)^2}$$

(14.4.26)

where

$$\beta = 2\pi N \langle v \rangle_{\text{rel}} \int_0^\infty \sin[\eta(\rho)]\, \rho\, d\rho$$

(14.4.27)

*Limits of Validity for Impact Phase-Shift Theory*    In developing the impact phase-shift theory, we tacitly assumed that the collisions were adiabatic. By that we mean that all the perturbing energy was contained in the perturbation and none was lost to other processes. There were no collisional transitions within the energy level or between the split levels of the degenerate cases. This will be a reasonable approximation as long as the splitting of the degenerate levels or the width of the perturbed level is greater than the uncertainty of energy of the perturber due to the Heisenberg uncertainty principle.

## Table 14.1 Types of Collisional Broadening

| | Type of Energy Level | |
| Type of Perturber | *Degenerate* | *Nondegenerate* |
| --- | --- | --- |
| Ion or electron | Linear Stark effect $n = 2$ | Quadratic Stark effect $n = 4$ |
| Neutral atom | Self-broadening $n = 3$ | van der Waal's broadening $n = 6$ |

Since the duration of the collision is of the order of $\rho/v$, the uncertainty of the colliding particle's energy is of the order

$$\Delta E \frac{\rho}{v} \simeq \hbar$$

(14.4.28)

In equation (14.4.16) we estimated the energy of the perturbation itself so that

$$\frac{v\hbar}{\rho} < \Delta\omega\,\hbar = \frac{2\pi C_n\hbar}{r^n} \simeq \frac{2\pi C_n\hbar}{\rho^n}$$

(14.4.29)

which requires that

$$\rho < \left(\frac{2\pi C_n}{v}\right)^{1/(n-1)} \simeq \rho_0$$

(14.4.30)

So it appears that only collisions that occur inside the Weisskopf radius will be adiabatic, and all the energy of the collision goes into perturbing the energy level.

However, if the impact parameter is too small, the classical path approximation will be violated and the duration of the collision will exceed the radiation time [equation (14.4.1)]. The extent of this constraint can be estimated by

noting that the duration of the collision is of the order of $\rho/v$. The radiation time is of the order of $1/\Delta\omega$, so that

$$\frac{\rho\,\Delta\omega}{v} \ll 1$$

(14.4.31)

if the classical path approximation is to be valid. However, equation (14.4.29) places a constraint of the impact parameter $\rho$ that must be met if the collisions are to be adiabatic. Using equation (14.4.29) to eliminate $\rho/v$ from equation (14.4.31) we get

$$\Delta\omega \ll \frac{v^{n/(n-1)}}{(2\pi C_n)^{1/(n-1)}}$$

(14.4.32)

Obviously the impact phase-shift theory will be valid only for the inner part of the line. For the outer part we must turn to another description of collisional broadening.

### b        Static (Statistical) Broadening Theory

In some real sense, the impact phase-shift theory follows the life history of a single radiating (or absorbing) atom which is subject to numerous weak collisions of short duration. The atomic absorption coefficient is then represented by the average of many atoms in various phases of that temporal history. In static broadening theory, the atomic absorption coefficient is constructed from the average of many atoms that are subject to the electric field of perturbers scattered randomly about. The opposite assumption is made concerning the duration of the collision compared to the radiation time. That is, the collision time is much longer than the radiation time, so that

$$t_{col} \gg t_{rad}$$

(14.4.33)

It is as if we took a picture of the perturbed atom with a shutter duration of the radiation time for the photon. In the impact phase-shift theory, we would see a blur of colliding tracks of the perturbers, while in the case of statistical broadening the picture would show individual perturbers fixed in space and some might be quite close to the atom in question. We are most interested in these near perturbers, for they are responsible for the largest perturbations to the atomic energy levels which in turn generate the broadest part of the line. This is precisely the part of the line for which the impact phase-shift theory fails.
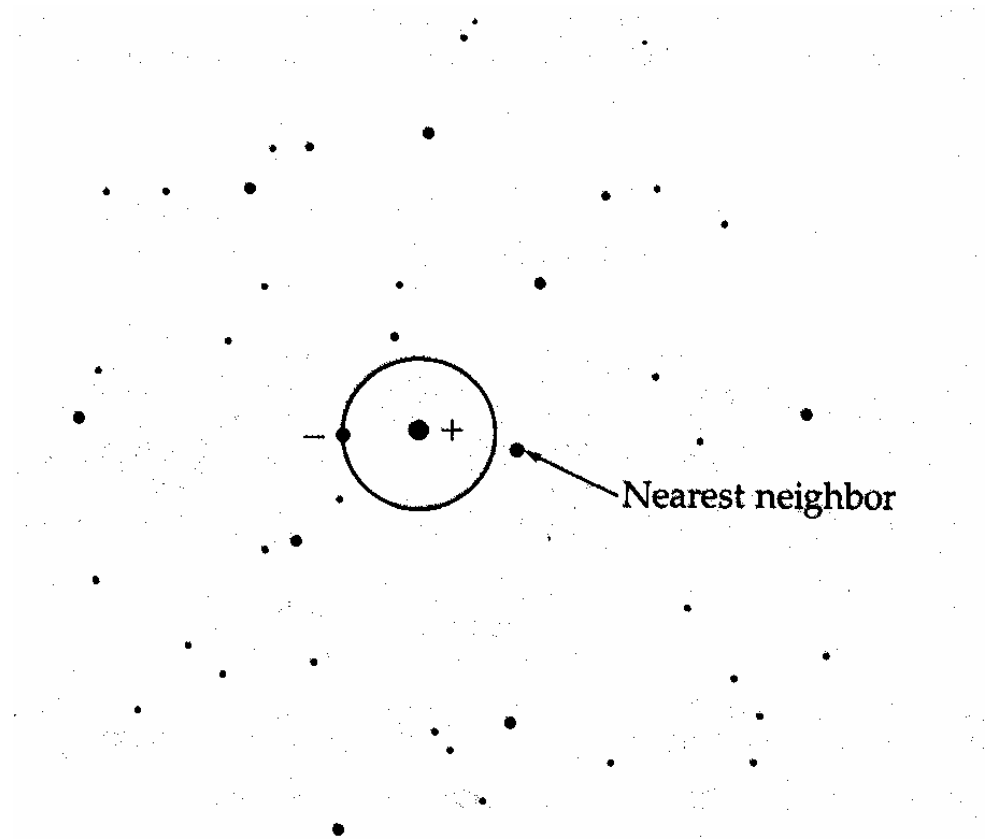
.

**Figure 14.5** shows a schematic view of the universe of perturbers under the assumptions of the Static Theory of broadening. The perturbers are randomly distributed in space, but only the "nearest neighbor" will be used for calculating the perturbing electric field.

Remember that the perturbation arises from the presence of an external electric field. In the static theory of line broadening, all particles are fixed in space, and the perturbing electric field is the vector sum of the electric fields of all the perturbers (see Figure 14.5). However, since we are concerned mostly about the strongest perturbations that form the wings of the line, we address only the perturbers closest to the atom in question

***Nearest-Neighbor Approximation and the Distribution of Electric Fields***
The assumptions required for the development of the static theory of broadening are similar in form and content to those required of the impact phase-shift theory. The collision time is assumed to be much larger than the radiation time [equation (14.4.33)] so that from the point of view of the radiating atom, the universe is frozen in time. Implicit in this assumption is the notion that every perturber has a well-defined position (zero) and momentum with regard to the perturbed atom. This is

equivalent to the classical path approximation of the impact phase-shift theory in that the position and momentum of the perturber are specified throughout the interaction time and are such that they are unaffected by the interaction.

To these complementary assumptions we add one more. Let us assume that the perturbative electric field can be represented by the electric field of the perturber closest to the atom and by that perturber alone. This is known as the *nearest-neighbor approximation*. Our task, then, is to find the probability distribution function for the perturber lying within a specified distance and thereby producing a perturbing electric field of a particular strength. Consider a spherical shell of thickness *dr* located a distance r from the perturbed atom (see Figure 14.5), and let the probability that the *nearest neighbor* is located within that shell be *P(r)dr*. Then the probability that the nearest neighbor lies within a sphere of radius *r* is $\int_0^r P(r)dr$.

Since the universe is not empty, there must be a nearest neighbor somewhere, so that the probability that the nearest neighbor does not lie within that sphere is $(1 - \int_0^r P(r)dr)$. Now if the region around the perturbed atom is of uniform density, the probability of finding any perturber within the spherical shell of thickness *dr* located at *r* is $4\pi r^2 ndr$, where *n* is the perturber density. Thus, the probability that the particle in that shell is the nearest neighbor is just the probability that there is a particle there multiplied by the probability that there is no particle nearer to the perturbed atom. So

$$P(r)\,dr = (4\pi r^2 n\,dr)\left[ 1 - \int_0^r P(r)\,dr \right]$$

(14.4.34)

This is an integral equation for the distribution function of nearest neighbors *P(r)*. We can solve it most easily by differentiating with respect to *r* and forming a differential equation for, $P(r)/4\pi r^2 n$. The solution to this equation is

$$P(r)\,dr = 4\pi r^2 n e^{-4\pi r^3 n/3}\,dr$$

(14.4.35)

However, we need the probability distribution of perturbing electric fields, so we assume that the perturber has a field that behaves as

$$E = \frac{a}{r^m}$$

(14.4.36)

Then, by substituting this dependence of the electric field on *r* into equation (14.4.35), the probability that an atom will see a perturbing electric field of strength E is

$$W_m(E)\,dE = \frac{3}{m}\,\frac{E_0^{3/m}}{E^{(m+3)/m}}\,e^{-(E_0/E)^{3/m}}\,dE$$

(14.4.37)

where

$$E_0 = a\left(\frac{4\pi n}{3}\right)^{m/3}$$

(14.4.38)

and it is sometimes called the *normalizing field strength*. If we further define the dimensionless quantity

$$\beta = \frac{E}{E_0}$$

(14.4.39)

we can write the probability distribution for this dimensionless field strength as

$$W_m(\beta)\,d\beta = \frac{3}{m}\,\beta^{-(m+3)/m}e^{-\beta^{-3/m}}\,d\beta$$

(14.4.40)

Finally, if we consider the case for broadening by ions or electrons, then $m = 2$ and we have

$$W_2(\beta)\,d\beta = \frac{3}{2}\,\beta^{-5/2}e^{-\beta^{-3/2}}\,d\beta$$

(14.4.41)

which is usually called the *Holtsmark distribution function*. As can be seen from Figure 14.6, the probability of finding a weak field due to the nearest neighbor is very small simply because it is unlikely that the nearest neighbor can be so far away and still be the nearest neighbor. As the field strength rises, so does the probability of it being the perturbing field, peaking between 1 and 2 times the normalized field strength. Stronger fields become less likely because the volume of space surrounding the atom within which the perturber would have to exist becomes just too small.

**Behavior of the Atomic Line Absorption Coefficient** If we assume that the perturbative change in the atomic energy level is proportional to the electric field to some power, then we can write

$$\Delta E = \Delta(h\nu) = h\,\Delta\nu \propto E^q \propto \beta^q$$

(14.4.42)

We can then use the nearest-neighbor distribution function to generate a probability density distribution function for the absorption of photons at a particular frequency shift $\Delta\nu$ as

$$P(\Delta\nu)\,d\nu \propto \Delta\nu^{-[1+3/(mq)]}e^{-\Delta\nu^{-3/(mq)}}\,d\nu$$

(14.4.43)

For large frequency or wavelength shifts, the argument of the exponential approaches zero, so the wavelength-dependent probability of absorption becomes

$$P(\Delta\lambda)\,d\lambda \propto \Delta\lambda^{-[1+3/(mq)]}$$
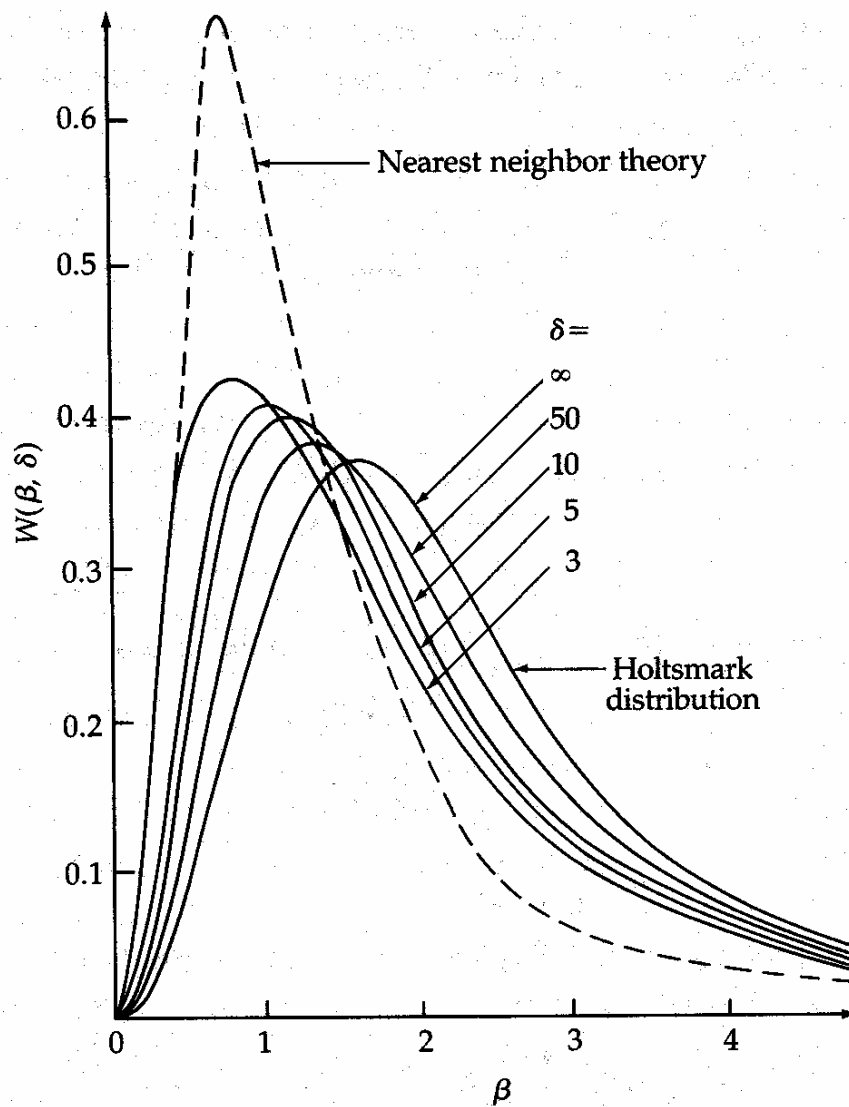
(14.4.44)

**Figure 14.6** shows the Nearest Neighbor distribution function for the perturbing electric field of the nearest neighbor assuming that it is an ion or electron as the solid line. The dashed line is for the Holtsmark distribution that includes the contribution from the rest of the gas. The parameter $\delta$ is a measure of the screening potential of the nearest neighbor [see Mihalas[11] (pp. 292-295)].

This is precisely the range for which the static theory through the nearest-neighbor approximation was expected to be accurate. Since the atomic line absorption coefficient is indeed proportional to the probability of photon absorption, its behavior in the far wings of a line is

$$S_{\Delta\lambda} \propto \Delta\lambda^{-[1+3/(mq)]}$$

(14.4.45)

Table 14.2 provides a brief summary of the asymptotic dependence of the absorption coefficient in the wings of the line for the various types of interactions discussed.

**Table 14.2 Asymptotic Behavior of the Atomic Line Absorption Coefficient**

| Type of Perturber | Type of Energy Level | |
|---|---|---|
| | *Degenerate q = 1* | *Nondegenerate q = 2* |
| Ion or electron $m = 2$ | Linear Stark effect $S_\lambda \sim \Delta\lambda^{-5/2}$ | Quadratic Stark effect $S_\lambda \sim \Delta\lambda^{-7/4}$ |
| Neutral atom $m = 3$ | Self-broadening $S_\lambda \sim \Delta\lambda^{-2}$ | van der Waal's broadening $S_\lambda \sim \Delta\lambda^{-3/2}$ |

Finally, we may find the constant of proportionality for the atomic line absorption coefficient in terms of the interaction constant for the force law $C_l$. This is analogous to the constant $C_n$ that appears in equation (14.4.17) and is usually determined empirically. In terms of this constant, the atomic line absorption coefficient becomes

$$S_\nu = \frac{3C_l}{(4\pi N/3)^{1/3} a_m} \Delta\nu^{-(1+3/l)} e^{-\Delta\nu^{-3/l}}$$

(14.4.46)

where $a_m$ is given by equation (14.4.20), and

$$l \equiv mq$$

(14.4.47)

In the broadening of degenerate levels, the splitting of the energy levels is so large that the line should be considered to consist of individual linear Stark components, each of which is quadratically Stark broadened. Under these conditions, the atomic line absorption coefficient for the combined Stark components becomes

$$S_\nu = \sum_k \frac{3C_{kl}}{(4\pi N/3)^{1/3} a_m} \Delta\nu_k^{-(1+3/l)} e^{-\Delta\nu_k^{3/l}}$$

(14.4.48)

If one were to improve on the static theory, the most obvious place would be to relax the nearest-neighbor approximation. The problem of including an ensemble of perturbers, all with their electric fields adding vectorially, was considered by J.Holtsmark[22] and solved by S. Chandrasekhar[23], who also provides tables of the results. As one might expect, the resultant form is similar to that of the nearest-

neighbor distribution in shape but somewhat more spread out (see Figure 14.6). Unfortunately the result takes the form of an integral so a complete description must be obtained numerically. However, for β in the vicinity of 1 we get the following asymptotic formula for $W(\beta)$:

$$W(\beta) = 1.496\beta^{-5/2}(1 + 5.107\beta^{-3/2} + 14.43\beta^{-3} + \cdots +)$$

(14.4.49)

As we might expect, the lead term of this series is just that of the nearest-neighbor approximation [see equation (14.4.41).]

### Limits of Validity and Further Improvements for the Static Theory

Since the assumption relating the collision time to the radiation time led to a limit on the range of validity for the impact phase-shift theory [equation (14.4.32)], we should not be surprised if the same were true for the static theory. This is indeed the case and the result is known as the *Holstein relation*, can be deduced from equation (14.4.32) almost by inspection:

$$\Delta\omega \gg \frac{v^{n/(n-1)}}{(2\pi C_n)^{1/(n-1)}}$$

(14.4.50)

So, as we hoped at the outset of the development of the static theory, it will be valid for precisely those regions of the line profile for which the impact phase-shift theory fails.

Of course, any microscopic inspection of a problem usually finds phenomena that provide additional complications for the solution. For example, we have assumed that the perturbers interact with the atom in question but do not interact among themselves. In reality an ion will attract electrons so as to create a neutral plasma on as small a scale as possible. In effect, then, the plasma will try to shield the ions from even more distant perturbers. This phenomenon is known as *Debye shielding* and is discussed in some detail by Mihalas[11] (pp. 292-295). The basic effect is density dependent and tends to flatten the Holtsmark distribution still further, thereby broadening the line even more. Fortunately, for normal stellar atmospheres the densities are not large enough to make Debye shielding a major effect until one reaches optical depths in the line that are quite remote from the boundary.

The treatment of collisional line broadening described so far has been based on purely classical considerations and has now been largely replaced by quantum mechanical calculations of the atomic line absorption coefficient for the more important stellar spectral lines. However, the quantum mechanical treatment is considerably less transparent than the classical one, so we give only the basic form. The power spectrum for the line is given by

$$I(\omega) = \frac{2\omega^4}{3\pi c^2} \, Tr\left[ \mathbf{P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{Q}(t)e^{-i\omega t}\mathbf{Q}(t')e^{i\omega t'} \, dt \, dt' \right]$$

(14.4.51)

where **P** is the probability density matrix for the atomic states involved in the formation of the spectral line, and **Q** is the matrix of dipole moments with elements

$$Q_{ij} = \int \Psi_i^* q \Psi_j \, dV$$

(14.4.52)

The wave functions $\Psi_i$ must include the effects of the perturbers as well as the atomic states of interest. A very complete discussion of the quantum theory of spectral line formation is given by Hans Griem[24].

For simple lines the classical theories of collisional broadening produce line profiles that agree well with observation. However, for the stronger lines of hydrogen and helium, any serious model should involve an atomic absorption coefficient based on the quantum mechanical description. While tables of these coefficients exist for many important lines (see Griem[24] and references there), much remains to be done to produce accurate values for many lines of astrophysical interest.

## 14.5   Curve of Growth of the Equivalent Width

While we have discussed the most important aspects of the formation of spectral lines, we have said little about the most important contributor to the appearance of the line in the spectrum - the abundance of the atomic species giving rise to the line. Obviously the more absorbers present in the atmosphere, the stronger the associated spectral line will appear. However, the quantitative relationship between the abundance and the equivalent width is not simple and is worthy of some discussion. Although most contemporary determinations of elemental abundances rely on detailed atmospheric modeling with the abundance as a parameter to be determined from comparison with observation, the classical picture of the relation between the equivalent width and the abundance is quite revealing about what to expect from such models. That classical quantitative relationship is known as the curve of growth. Some students have wondered what is growing in the curve of growth. The answer is that the equivalent width increases or "grows" with increasing abundance.

### a        Schuster-Schwarzschild Curve of Growth

To create a curve of growth, we must relate the equivalent width to the atomic abundance. This requires some model of the atmosphere in which the atoms reside.  For purposes of illustration, we take the simplest model possible. In Chapter 13 we set up the equation of radiative transfer for line radiation [equation (13.1.6)], and we solved it for some special cases. For the Schuster-Schwarzschild atmosphere, this led to a line profile given by equation (13.2.8):

$$r_v = \frac{F_v}{F_c} = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1}$$

(14.5.1)

The definition of $\tau_0$ allows us to write

$$\tau_0 = \int_0^{\tau_0} dt_v = \int_0^{x_0} \kappa_v \rho\, dx = \int_0^{x_0} n_i S_v\, dx = \langle S_v \rangle \int_0^{x_0} n_i\, dx = N_i \langle S_v \rangle$$

(14.5.2)

where $N_i$ is the column density of the atom giving rise to the line and $\langle S_v \rangle$ is the line absorption coefficient averaged over depth. Since for this simple model the atmospheric conditions are considered constant throughout the cool gas, we drop the average-value symbols for the remainder of this section. We have already seen [equation (14.3.27)] that for many atomic lines the atomic line absorption coefficient, including the effects of radiation damping, collisional damping, and Doppler broadening, can be written as

$$S_v = S_0 H(a,\, u)$$

(14.5.3)

where *H(a,u)* can be either the Voigt or normalized Voigt function depending on what constants have been absorbed into $S_0$. Thus the line profile for the Schuster-Schwarzschild atmosphere is

$$r_v = \left(1 + \frac{\sqrt{3}\tau_0}{2}\right)^{-1} = \left[1 + \frac{\sqrt{3}S_0 H(a,\, u)N_i}{2}\right]^{-1}$$

(14.5.4)

To relate this to the equivalent width, equation (14.5.4) must be integrated over the frequencies contained the line so that

$$W_\lambda = 2\int_0^\infty \frac{\sqrt{3}\tau_0/2}{1 + \sqrt{3}\tau_0/2}\, d\lambda = 2\int_0^\infty \frac{[\sqrt{3}S_0 H(a,\, u)N_i/2]\, d\lambda}{1 + \sqrt{3}S_0 H(a,\, u)N_i/2}$$

(14.5.5)

It is convenient to express the frequency-dependent optical depth in the line in terms of the optical depth at the line center $\chi_0$ so that

$$\tau_0(v) = \frac{\chi_0 H(a,\, u)}{H(a,\, 0)}$$

(14.5.6)

From equations (14.5.2), and (14.5.3)

$$\chi_0 = S_0 H(a,\, 0)N_i \approx \frac{\sqrt{\pi}e^2 f_{jk}N_i\lambda_0}{m_e v_0 c}$$

(14.5.7)

Consider the case where the damping constant is small compared to Doppler broadening so that $a < 0.2$. Then the Doppler core will dominate the line profile, and

we can write the optical depth in the line as

$$\tau_0(\Delta\lambda) = \chi_0 e^{-(\Delta\lambda/\Delta\lambda_d)^2} = \chi_0 e^{-\xi^2}$$

(14.5.8)

where $\xi \equiv \Delta\lambda/\Delta\lambda_d$. Substitution into equation (14.5.5) yields

$$W_\lambda = \sqrt{3}\,\Delta\lambda_d\,\chi_0 \int_0^\infty \left(e^{\xi^2} + \frac{\sqrt{3}\chi_0}{2}\right)^{-1} d\xi$$

(14.5.9)

The integral can be expanded in a series so that

$$\int_0^\infty \left(e^{\xi^2} + \frac{\sqrt{3}\chi_0}{2}\right)^{-1} d\xi \approx \int_0^\infty \left[e^{-\xi^2} - \frac{\sqrt{3}\chi_0}{2}e^{-2\xi^2} + \frac{3\chi_0^2}{4}e^{-3\xi^2}\right.$$

$$\left. + \cdots + (-1)^k \left(\frac{\sqrt{3}}{2}\right)^k \chi_0^k e^{-(k+1)\xi^2}\right]d\xi$$

(14.5.10)

But

$$\int_0^\infty e^{-k\xi^2}\,d\xi = \frac{1}{2}\sqrt{\frac{\pi}{k}}$$

(14.5.11)

Thus, we can write the equivalent width in the line as

$$W_\lambda = \frac{\sqrt{3\pi}}{2}\chi_0\,\Delta\lambda_d\left[1 - \chi_0\sqrt{\tfrac{3}{8}} + \frac{\chi_0^2\sqrt{3}}{4} - \frac{9\chi_0^3}{8} + \cdots\right]$$

(14.5.12)

This, then, represents the first part of the curve of growth, and the equivalent width is indeed directly proportional to $\chi_0$ and hence the abundance $N_i$. This is a commonsense result that simply says that the number of photons removed from the beam is proportional to the number of atoms doing the absorbing, so that section of the curve of growth is known as the *linear* section.

However, the seeds of difficulties are apparent in the higher-order terms in equation (14.5.12). As the number of absorbers increases, we would expect that some atoms high in the atmosphere to be "shadowed" by atoms lower in the atmosphere. When all the photons at a given frequency have been absorbed, then the further addition of atoms that can absorb at those frequencies will make no change in the equivalent width. When this happens, the line is said to be saturated. As the optical depth in the line center $\chi_0$ increases, the term in brackets will fall below unity and the curve of growth will increase more slowly than the linear growth. For $0 \leq \chi_0 \leq 0.5$, the series may be terminated after the first term. However, for larger values of $\chi_0$, a somewhat different expression of the integral on the left hand side of equation (14.5.10) is in order. If we make the transformation

$$\frac{\sqrt{3}\chi_0}{2} = e^b \qquad \xi = \zeta^{1/2}$$

(14.5.13)

the equivalent width becomes

$$W_\lambda = \Delta\lambda_d \int_0^\infty \frac{\zeta^{-1/2}\,d\zeta}{1 + e^{\zeta - b}} = 2\,\Delta\lambda_d\sqrt{b}\left(1 - \frac{\pi^2}{24b^2} - \frac{\pi^4}{384b^4} - \cdots\right)$$

(14.5.14)

If $\chi_0 > 55$, all but the lead term of the approximation may be ignored. However, in the region where $0.5 < \chi_0 < 55$, the series given by either equation (14.5.12) or equation (14.5.14) must be used. From the lead term of equation (14.5.14) it is clear that as the Doppler core saturates, the equivalent width grows very slowly as

$$\frac{W_\lambda}{\lambda} \propto \sqrt{\ln N_i}$$

(14.5.15)

This is known as the "flat" part of the curve of growth.

As the abundance increases still further, a significant number of atoms will exist that can absorb in the damping wings of the line and the equivalent width will again begin to increase, but at a rate that will depend on the damping constant appropriate for the line (see Figure 14.7).

Once more we will need a different representation of the optical depth that is appropriate for the damping wings of the line. From the definition of the dimensionless variables of the Voigt function [see equation (14.3.23)]

$$dv = \Delta v_d\,du$$

(14.5.16)

so that we can rewrite equation (14.5.5) with the aid of equation (14.5.6) to obtain

$$W_v = 2\,\Delta v_d \int_0^\infty \frac{du}{1 + 2H(a, 0)/[\sqrt{3}\chi_0 H(a, u)]}$$

(14.5.17)

The Voigt function as given in equation (14.3.26) can be approximated for large u as

$$H(a, u) \approx \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{u^2}\,dy = \frac{a}{\sqrt{\pi u^2}}$$

(14.5.18)

For modest values of the damping parameter $a$, $H(a,0)$ is near unity so that

$$W_v \approx \left(\frac{\sqrt{2}\,\Delta v_d}{c}\right)(3^{1/4}\pi^{3/4}\sqrt{\chi_0 a})$$

(14.5.19)

So for large abundances the curve of growth will again increase in a manner that depends on the square root of the damping constant as well as the square root of the

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abundance. Except for the separation brought about by the growth of the damping wings of the line, the curve of growth is a single-valued function of $W_\lambda/\Delta\lambda_d$ versus the optical depth at the line center $\chi_0$. Both these parameters are dimensionless, so for this model a single curve satisfies all problems. However, it is worth remembering that the Schuster-Schwarzschild model is correct for scattering lines only, and very few spectral lines that go into abundance calculations are scattering lines. Thus, the classical curve of growth can give only very approximate results even if it is calculated exactly.
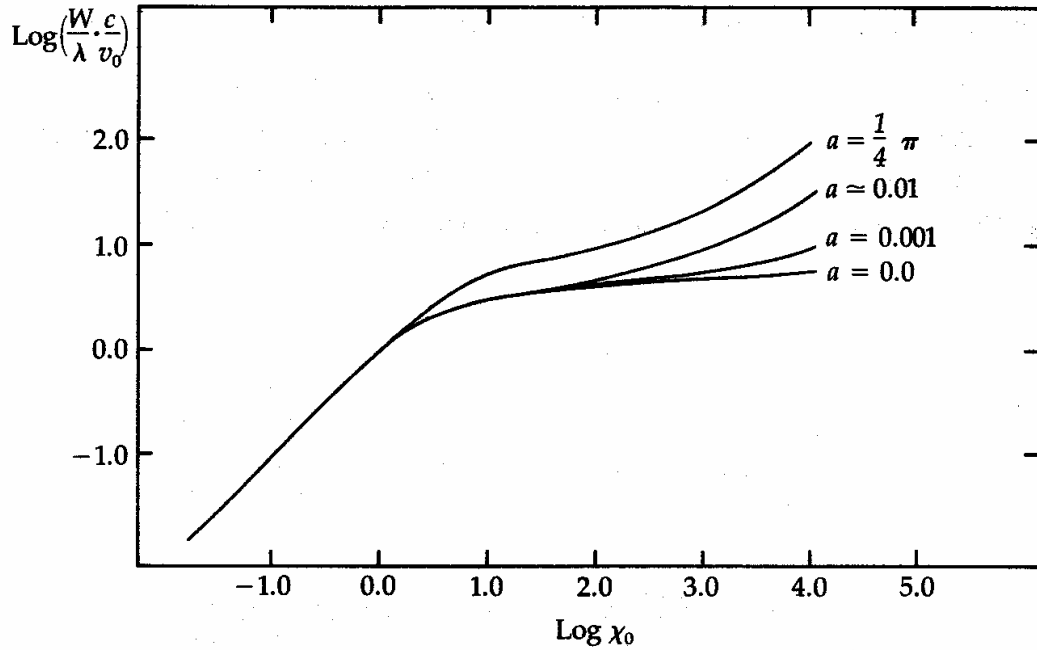


**Figure 14.7** shows the curve of growth for the classical Schuster-Schwarzschild model atmosphere.

### b      More Advanced Models for the Curve of Growth

There are several ways to improve the accuracy of the curve of growth. First, we could use a more accurate solution to the equation of radiative transfer such as the Chandrasekhar discrete ordinate method. The use of the equations of condition on the boundary values [equation (10.2.31)] enables us to obtain a profile of the form

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$$r_v = \frac{\displaystyle\sum_{\alpha=1}^{n-1} L_{c,+\alpha} + Q_c}{\displaystyle\sum_{\alpha=1}^{n-1} L_{+\alpha}e^{-k_\alpha \tau_0(v)} + L_{-\alpha}e^{+k_\alpha \tau_0(v)} + \tau_0(v) + Q}$$

(14.5.20)

The behavior of the optical depth could then be substituted into equation (14.5.20) and from there into equation (14.5.5), thereby relating the equivalent width to the optical depth at the line center. However, this would only improve the details of the radiative transfer without improving the model itself. Since we know that the errors of the two-stream (Eddington) approximation are of the order of 12 percent, this is a small improvement indeed for the additional work involved.

A significant improvement could be made by using the Milne-Eddington model atmosphere. Here the line profile is given by equation (13.2.29), where the frequency dependence is entirely contained in the behavior of $\mathcal{L}_v$, $\varepsilon_v$, and $\eta_v$ with frequency. In addition, the parameters *a* and *b* which describe the surface temperature and temperature gradient need to be specified. Laborious as the task of constructing these more sophisticated curves of growth is, it was done by Marshall Wrubel[25-27] in a series of papers. Although the additional parameters required by the model are annoying, the improvement in the representation of the star by these models is usually worth the effort. It is probably not worth the trouble to generate more sophisticated classical models than these. Direct modeling by a model atmosphere code is the appropriate approach, for one can remove virtually all the assumptions required for the classical models so that the accuracy is largely determined by the accuracy of the atomic constants characterizing the line.

### c    Uses of the Curve of Growth

***Determination of Doppler Velocity and Abundance***    We already indicated that the curve of growth can be used to estimate stellar abundances. However, it is possible (in principle) to learn a great deal more about the conditions in the atmosphere of the star from the curve of growth. Imagine that we have measured equivalent widths for a collection of lines that all arise from the same lower level for which the atomic parameters and damping constants are accurately known. Further suppose that the values for the lines span a reasonable range of the curve of growth. Thus we have empirical values for $W(\lambda_i)/\lambda_i$ and $\sqrt{\pi}e^2 f_i \lambda_i /(m_e c)$. The second of these two quantities, which we call $X_i$ is given by

$$\mathrm{Log}\, X_i = \mathrm{Log}\, \chi_0 + \mathrm{Log}\, \frac{v_0}{N}$$

(14.5.21)

Thus, a plot of Log $X_i$ versus. Log$[W(\lambda_i)/\lambda_i]$ will yield an empirical curve of growth that differs from the theoretical curve given in Figure 14.7 by a shift in both the ordinate and the abscissa. Since the lines all arise from the same lower level, N is the same for all points. The horizontal shift then specifies Log($v_0$/N), while the vertical shift specifies Log($c/v_0$). Thus both the abundance and the Doppler velocity are determined independently. To the extent that the kinetic temperature is known, we know the microturbulent velocity. If the span of the curve of growth is large enough to determine *a*, an average value of $\Gamma_c$ may also be found.

### *Determination of the Excitation Temperature*

Consider the situation where, in addition to the information given above, we know the equivalent widths for a number of lines arising from different states of excitation. Further assume that LTE holds so that the populations of those excited states are given by the Boltzmann formula. Then

$$\text{Log } X_i = \text{Log } \chi_0 + \text{Log } \frac{v_0}{N} - \text{Log } \frac{g_i e^{-\epsilon_i/(kT)}}{U(T)}$$

(14.5.22)

We have already determined $v_0$, so we may correct the observed equivalent widths so that the observed values are brought into correspondence with the theoretical ordinate of the curve of growth $W_\lambda/\Delta\lambda_d$. The horizontal points will now miss the theoretical curve of growth by an amount

$$\aleph_i = \text{Log } X_i - \left( \text{Log } \chi_0 + \text{Log } \frac{v_0}{N} \right) + \text{Log } \frac{g_i e^{-\epsilon_i/(kT)}}{U(T)}$$

(14.5.23)

or

$$\text{Log } \frac{g_i e^{-\epsilon_i/(kT)}}{U(T)} + \text{const} = \aleph_i$$

(14.5.24)

Since everything about the lines in equation (14.5.24) is known, only the constant and the temperature are unknowns, and they can be determined by least squares.

Important parameters concerning the structure of a stellar atmosphere can be estimated from the classical curve of growth. Not only can the abundance of the elements that make up the atmosphere be measured, but also the turbulent velocity and excitation temperature can be roughly determined. However, to use the classical curve of growth is to make some very restrictive assumptions. The assumption that the parameters determining the lines are independent of optical depth is a poor assumption and is usually the reason that the excitation temperature does not agree with the effective temperature. In addition, the thickness of the atmosphere is probably not the same for all the lines used. Finally, the lines are usually not scattering lines. Nevertheless, the method should be used prior to undertaking any detailed analysis in order to set the ranges for the expected solution. Any sophisticated analysis that produces answers wildly different from those of the curve

of growth should be regarded with suspicion.

Finally, sooner or later, we must be wary of the assumption of LTE. In the upper layers of the atmosphere, the density will become low enough that collisions will no longer occur frequently enough to overcome the nonequilibrium effects of the radiation field, and the level populations of the various atomic states will depart from that given by the Saha-Boltzmann ionization-excitation formula. This will particularly affect the strong spectral lines that are formed very high up in the atmosphere. In the next chapter, we survey what is to be done when LTE fails.

## Problems

1.  Imagine a line whose intensity profile is

$$f_v = 1 + \delta(v - v_0)$$

Calculate the observed line profile for a radially expanding atmosphere which exhibits a velocity gradient

$$\frac{dv}{d\tau_c} = \text{const} = c\frac{\Delta\lambda_0}{\lambda_0}$$

State any assumptions that you make in solving the problem.

2.  Consider a line generated by atoms constrained to move perpendicular to a radius vector from the center of the star. Find an expression for the atomic absorption coefficient due to Doppler broadening alone.

3.  Find the natural width for
    a     H$\beta$
    b     Mg II ($\lambda4481$)
    c     FeI ($\lambda3720$) .

4.  Estimate the transition times from the natural widths of the lines in Problem 3, and compare them with a crude estimate of the collision rates for atoms in these states. State clearly any assumptions you make. In what kind of star would you expect to find these spectral lines?

5.  If both the atoms of a radiating gas and the particles perturbing them are in statistical equilibrium, show that the average relative velocity between them is given by

$$\langle v \rangle = \left[ \left( \frac{8kT}{\pi\mu_h} \right) \left( \frac{1}{A_1} + \frac{1}{A_2} \right) \right]^{1/2}$$

where $\mu_h$ = the mass of a unit atomic weight and $A_1$ and $A_2$ are the atomic weight of the atom and perturber respectively.

6.  Find the far-wing dependence of the line absorption coefficient of an atom having nondegenerate energy levels which are broadened by perturbers having only octopole moments of their charge configurations.

7.  Compute a line profile for Si II($\lambda\lambda 6347.10$) for an A0V star. Use a model atmosphere code if possible.

8.  Show that $W_\lambda/\lambda = W_v/v$

9.  Use a model atmosphere code such as ATLAS to generate "curves of growth" for Fe I($\lambda\lambda 4476$), Mg II($\lambda\lambda 4481$), and Si II($\lambda\lambda 4130$). Include a microturbulent velocity of 2 km/s. Consider the reference atmosphere to be one with $T_e = 10^4$K, Log $g = 4.0$, and solar abundance (except for Fe, Mg, and Si). Compare your results with the classical curve of growth for a Schuster-Schwarzschild model atmosphere and obtain values for $\Delta\lambda_d$, the kinetic temperature, microturbulent velocity, and $\Gamma$ for each line. Compare your results with the values used to generate the line profiles and discuss any differences.

10. Consider the following situation: A 1-mm beam of neutral hydrogen gas with an internal kinetic temperature of $10^4$ K is accelerated to an energy of $10^{-3}$eV per atom. The beam enters a 10-m vacuum chamber and is directed toward a 1-cm bar located in the center of the chamber and oriented at right angles to the beam. The bar has been charged to $10^7$v. The beam passes through a 1-mm hole in the bar and proceeds out the opposite side of the chamber. A spectrograph is placed so that it "looks" along the beam and sees the beam against a $2 \times 10^4$ K continuum blackbody source located near where the beam enters the chamber. Assuming that the beam density is sufficiently low to ensure that it is optically thin, but high enough to establish LTE, find the line profile for Hβ. Further assume that the central depth of the line is 0.6. Find the equivalent width of Hβ and the density of hydrogen. On the basis of your results, discuss the validity of the assumptions you used.

11. Consider a Schuster-Schwarzschild model atmosphere populated with several types of atoms having different atomic absorption coefficients. Find the theoretical curves of growth for each of these atoms.

(a) $S_1(v) = a + \dfrac{b}{|\Delta v|}$

(b) $S_2(v) = \delta(\Delta v)$

(c) $S_3(v) = \begin{pmatrix} a & \text{for } |\Delta v| \leq \Delta v_0 \\ 0 & \text{for } |\Delta v| > \Delta v_0 \end{pmatrix}$

(d) $S_4(v) = \begin{pmatrix} a + b(\Delta v)^2 & \text{for } |\Delta v| \leq \left(\dfrac{-a}{b}\right)^{1/2} \\ 0 & \text{for } |\Delta v| > \left(\dfrac{-a}{b}\right)^{1/2} \end{pmatrix} \quad b < 0$

(e) $S_5(v) = \begin{pmatrix} a + b|\Delta v| & \text{for } |\Delta v| \leq \dfrac{-a}{b} \\ 0 & \text{for } |\Delta v| > \dfrac{-a}{b} \end{pmatrix} \quad b < 0$

Compare with the classical solutions for the curve of growth.

12. Let the probability of finding a value of the turbulent velocity projected along the line of sight v be uniform in the range $-v_0 \# v \# v_0$. The probability of finding a value of v outside this range is zero. In addition to turbulence, there are thermal Doppler motions present which correspond to a temperature T. Assuming that $f$ and $\Gamma$ are known, derive an expression for the atomic line absorption coefficient. Leave your answer in the form of a definite integral containing an error function.

13. Consider a certain atom in the solar atmosphere at a point where the hydrogen abundance $N_h = 10^{17} \text{cm}^{-3}$ and T = 5500 K. The atom has a strong resonance line at l=5000Å with an Einstein A coefficient of $9.7 \times 10^7 \text{s}^{-1}$. The atom has interacted with a neutral hydrogen atom so that a frequency shift of $\Delta\omega = 2 \times 10^6/r^2 \text{ s}^{-1}$ of the line frequency has resulted. Here, r is in angstroms.

   a  Make a reasonable estimate of how long the collision lasts.

   b  Qualitatively justify the type of broadening theory you would use to describe the atomic absorption coefficient.

   c  What is the approximate cross-section for this event?

   d  What is the value of $\Gamma$ you would obtain from the impact phase-shift theory of line broadening?

14. Suppose the data below are observed in a certain star. They all pertain to the lines of the neutral state of the same element which has a partition function of 2.0. The parameter $\varepsilon_i$ refers to the lower level of the transition.

| $j$ | $\lambda$, Å | $\epsilon_i$, eV | $g_i$ | $W_\lambda$, Å | $f_{ij}$ |
|---|---|---|---|---|---|
| 1 | 4,930 | 2.70 | 3 | 0.330 | 0.50 |
| 2 | 3,470 | 2.70 | 3 | 0.0155 | 0.050 |
| 3 | 5,676 | 2.70 | 3 | 0.0562 | 0.0667 |
| 4 | 3,360 | 1.80 | 6 | 0.0190 | 0.00316 |
| 5 | 10,050 | 1.80 | 6 | 1.349 | 1.060 |
| 6 | 3,862 | 1.80 | 6 | 0.0617 | ---- |

Using the Schuster-Schwarzschild model atmosphere, find

    a        the number of atoms per square centimeter above the photosphere,

    b        the missing *f* value, and

    c        the value for the Doppler velocity $v_0$.

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For a somewhat different approach to the problem of line broadening, the interested student should consult

Jefferies, J.T.:*Spectral Line Formation*, Blaisdell, New York, 1968, pp. 46 - 91.

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