6

Relativistic Stellar Structure

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In the next two chapters we consider some specific problems which lie outside the realm of normal stellar structure. In the past several decades, it has become increasingly clear that a large number of stars require some further subtleties of physics for their proper description. Two areas that we shall consider involve the initial assumption of spherical symmetry and the assumption that the gravitational field can be described by the Newtonian theory of gravity with sufficient accuracy to properly represent the star. In this chapter, we investigate some of the ramifications of the general theory of relativity for highly condensed objects and super-massive stars.

Although the application of the general theory of relativity to astronomical problems has a long and venerable history dating back to Einstein himself, it was not until the discovery of pulsars in the 1960s that a great deal of interest was directed toward the impact of the theory on stellar structure. To be sure, the pioneering theoretical work was done 30 years earlier and can be traced back to Landau[1] in 1932. The fundamental work of Oppenheimer[2,3] and collaborators still provides the fundamental basis for most models requiring general relativity for their representation. But it was the discovery that neutron stars actually existed and were probably the result of the dynamical collapse of a supernova that led to the construction of modern models that represent our contemporary view of these objects.

It is not my intent to provide a complete description of the general theory of relativity in order that the reader is able to understand all the ramifications for stellar structure implied by that theory. For that, the reader is referred to "*Gravitation*" by Misner, Thorne, and Wheeler[4]. Rather, let us outline the origin of the fundamental equations of relativistic stellar structure and the results of their applications to some simple objects, without the rigors of their complete derivation. The intent here is to provide some physical insight into the role played by general relativity in a variety of objects for which that role is important.

6.1 Field Equations of the General Theory of Relativity

The general theory of relativity is a classical field theory of gravitation in which all variables are assumed to be continuous and are uniquely specified. Thus, the Heisenberg uncertainty principle and quantum mechanics play no direct role in the theory. Although it is traditional to present general relativity in a system of units where $c = h = G = 1$, we adopt the nontraditional notion of generally maintaining the physical constants in the expressions in the hopes that the physical interpretation of the various terms may be clearer to the readers. However, we adopt the Einstein summation convention where repeated indices are summation indices for this section, to avoid the host of summation signs that would otherwise accompany the tensor calculus.

The basic philosophy of general relativity is to relate the geometry of space-time, which determines the motion of matter, to the density of matter-energy, known as the *stress energy tensor*. This relation is accomplished through the Einstein field equations. The geometry of space-time is dictated by the metric tensor which defines the properties of that geometry and basically describes how travel in one coordinate involves another coordinate, so that

$$ds^2 = \mathbf{g}_{\mu\nu}\, dx^\mu\, dx^\nu \tag{6.1.1}$$

The elements of the metric tensor are dimensionless; for ordinary Euclidean space they are all unity if $\mu = \nu$ and zero otherwise. If one were doing geometry on a deformed rubber sheet, this would not necessarily be true. In general, the distance traveled, expressed in terms of any set of local coordinates, will depend on the orientation of those coordinates on the rubber sheet. The coefficients that "weight" the role played by each coordinate in determining the distance according to equation (6.1.1), for all directions traveled, are the elements of the metric tensor. Now the field equations relate second derivatives of the metric tensor to the properties of the local matter-energy density expressed in terms of the stress-energy tensor. Specifically the Einstein field equations are

$$\mathbf{G}_{\mu\nu} = \frac{8\pi G}{c^2}\, \mathbf{T}_{\mu\nu} \tag{6.1.2}$$

Here $\mathbf{G}_{\mu\,\nu}$ is known as *the Einstein tensor* and $\mathbf{T}_{\mu\,\nu}$ is the stress energy tensor in physical units (say grams per cubic centimeter). The quantity G/c^2 is a very small number in any common system of units, which shows that the departure from Euclidean space is small unless the stress-energy is exceptionally large. The specific relation of the metric tensor to the Einstein tensor is extremely complicated and for completeness is given below.

Define

$$\Gamma_{\beta\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial \mathbf{g}_{\beta\nu}}{\partial x^\mu} + \frac{\partial \mathbf{g}_{\beta\mu}}{\partial x^\nu} + \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^\beta}\right) \tag{6.1.3}$$

and

$$\Gamma^\alpha{}_{\mu\nu} \equiv \mathbf{g}^{\alpha\beta}\Gamma_{\beta\mu\nu} \tag{6.1.4}$$

where $\mathbf{g}^{\alpha\beta}$ is the matrix inverse of $\mathbf{g}_{\alpha\beta}$. The symbol $\Gamma_{\beta\,\mu\,\nu}$ is known as the *Christoffel symbol*. The Christoffel symbols and their derivatives can be combined to produce the Riemann curvature tensor

$$\mathbf{R}^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \tag{6.1.5}$$

which when summed over two of its indices produces the Ricci tensor

$$\mathbf{R}_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} \tag{6.1.6}$$

This can be further summed (contracted) over the remaining two indices to yield a quantity known as the *scalar curvature*

$$R = \mathbf{R}^{\mu}{}_{\mu} \tag{6.1.7}$$

Finally, the Einstein tensor can be expressed in terms of the Ricci tensor, the scalar curvature, and the metric tensor itself as

$$\mathbf{G}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \mathbf{g}_{\mu\nu} R \tag{6.1.8}$$

For a given arbitrary metric, the calculations implied by equations (6.1.4) through (6.1.8) are extremely tedious, but conceptually simple. Since the metric tensor depends on only the geometry, and since the operations described in forming the Riemann and Ricci tensors, and scalar curvature are essentially geometric, nothing but geometry appears in the Einstein tensor. Hence the saying, "the left-hand side of the Einstein field equations is geometry, while the right-hand side is physics".

6.2 Oppenheimer-Volkoff Equation of Hydrostatic Equilibrium

a Schwarzschild Metric

For reasons that are obvious by now, much of the initial progress in general relativity was made by considering highly symmetric metrics which simplify the Einstein tensor. So let us consider the most general metric which exhibits spherical symmetry. This is certainly consistent with our original assumption of spherical stars. If we take the usual spherical coordinates r, θ, ϕ, and let *t* represent the time coordinate, then the distance between two points in this spherical metric can be written as

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2\,d\theta^2 - r^2(\mathrm{Sin}^2\,\theta)\,d\phi^2 + \frac{e^{\alpha(r)}\,dt^2}{c^2} \tag{6.2.1}$$

where λ(r) and α(r) are arbitrary functions of the radial coordinate r. We must also make some assumptions about the physics of the star in question. This amounts to specifying the stress energy tensor.

Consistent with our assumption of spherical symmetry, let us assume that the material of the star has an equation of state which exhibits no transverse strains, so that all the off-diagonal elements of the stress energy tensor are zero and the first three spatial elements are equal to the matter equivalent of the energy density. The fourth diagonal component is just the matter density so

$$T^{11} = T^{22} = T^{33} = -\frac{P}{3c^2} \qquad T^{44} = \rho \tag{6.2.2}$$

This is equivalent to saying that the equation of state has the familiar form

$$P = P(\rho) \tag{6.2.3}$$

Now if we take the metric tensor specified by equation (6.2.1), and go through the operations specified by equations (6.1.2) through (6.1.8), and sum over the three spatial indices because of the spherical symmetry, then the Einstein field equations become

$$e^{-\lambda}\left(\frac{\alpha'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^2}\frac{P}{c^2}$$
$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi G}{c^2}\rho \tag{6.2.4}$$

Here the prime denotes differentiation with respect to the radial coordinate r. This solution must hold through all space, including that outside the star where P = ρ = 0. If we take the boundary of the star to be where r = R, then for r > R we get the Schwarzschild metric equations

$$e^{-\lambda(r)}\left(\frac{1}{r}\frac{d\alpha(r)}{dr} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$$
$$e^{-\lambda(r)}\left(\frac{1}{r}\frac{d\lambda(r)}{dr} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \tag{6.2.5}$$

which have solutions

$$e^{-\lambda(r)} = 1 + \frac{A}{r} \qquad e^{-\alpha(r)} = B\left(1 + \frac{A}{r}\right) \tag{6.2.6}$$

where A and B are arbitrary constants of integration for the differential equations and are to be determined from the boundary conditions. At large values of r, we require that the metric go over to the spherical metric of Euclidean flat space, so that

$$\lim_{r\to\infty} e^{\lambda(r)} = \lim_{r\to\infty} e^{\alpha(r)} = 1 \tag{6.2.7}$$

and B = 1. A line integral around the object must yield a temporal period and distance consistent with Kepler's third law, meaning that A is related to the Newtonian mass of the object. Specifically,

$$A = -\frac{2GM}{c^2} \tag{6.2.8}$$

which has the units of a length and is known as the *Schwarzschild radius*.

b Gravitational Potential and Hydrostatic Equilibrium

Since

$$e^{\alpha(r)} \simeq 1 + \alpha(r) \simeq 1 + \frac{2GM}{c^2 r} \tag{6.2.9}$$

we know that

$$\alpha(r) = \frac{2\Omega}{c^2} \tag{6.2.10}$$

where **Ω** is the Newtonian potential at large distances. The parameter $\alpha(r)$ then plays the role of a potential throughout the entire Schwarzschild metric. So we can solve the first of equations (6.2.4) for its spatial derivative and get

$$\frac{d\Omega}{dr} = \frac{G[M(r) + 4\pi r^3 P/c^2]}{r[r - 2GM(r)/c^2]} \tag{6.2.11}$$

This is quite reminiscent of the Newtonian potential gradient, except (1) that the mass has been augmented by a term representing the local "mass" density attributable to the kinetic energy of the matter producing the pressure and (2) that the radial coordinate has been modified to account for the space curvature. Now even in a non-Euclidean metric we have the reasonable result

$$\nabla P = -\tilde{\rho}\nabla\Omega \tag{6.2.12}$$

where $\tilde{\rho}$ is the total local mass density so that the matter density, ρ, must be increased by P/c^2 to include the mass of the kinetic energy of the gas. [For a rigorous proof of this see Misner, Thorne, and Wheeler[4] (p601)]. Combining equations (6.2.11) and (6.2.12), we get

$$\frac{dP}{dr} = -\frac{G(\rho + P/c^2)[M(r) + 4\pi r^3 P/c^2]}{r[r - 2GM(r)/c^2]} \tag{6.2.13}$$

This is known as the Oppenheimer-Volkoff equation of hydrostatic equilibrium, and along with the equation of state it determines the structure of a relativistic star.

6.3 Equations of Relativistic Stellar Structure and Their Solutions

In many respects the construction of stellar models for relativistic stars is easier than that for Newtonian models. The reasons can be found in the very conditions which make consideration of general relativity important. Except in the case of super-massive stars, when gravity has been able to compress matter to such an extent that general relativity is necessary to describe the metric of the space occupied by the star, all forms of energy generation which might provide opposition to gravity have ceased. Because of the high degree of compaction, the material generally has a high conductivity and is isothermal, so its cooling rate is limited only by the ability of the surface to radiate energy. In addition, the high density leads to equations of state in which the kinetic energy of the gas is relatively unimportant in determining the state of the gas. The pressure is determined by inter-nuclear forces and thus depends on only the density. In a way, the messy detailed physics of low-density gas, which depends on its chemical composition and internal energy, has been "squeezed" out of it and replaced by a simpler environment where gravity rules supreme. To be sure, the equation of state of nuclear matter is still an area of intense research interest. But progress in this area is limited as much by our inability to test the results of theoretical predictions as by the theoretical difficulties themselves.

a A Comparison of Structure Equations

To see the sort of simplification that results from the effects of extreme gravity, let us compare the equations of stellar structure in the Newtonian limit, and the relativistic limit.

	Nonrelativistic		Relativistic
(a)	$\dfrac{dM(r)}{dr} = 4\pi r^2\rho$	Conservation of mass	$\dfrac{dM(r)}{dr} = 4\pi r^2\rho$
(b)	$\dfrac{d\Omega}{dr} = \dfrac{GM(r)}{r^2}$	Hydrostatic equilibrium	$\dfrac{d\Omega}{dr} = \dfrac{G[M(r) + 4\pi r^3 P/c^2]}{r[r - 2GM(r)/c^2]}$
(c)	$\dfrac{dP}{dr} = -\dfrac{\rho\, d\Omega}{dr}$		$\dfrac{dP}{dr} = -\left(\rho + \dfrac{P}{c^2}\right)\dfrac{d\Omega}{dr}$
(d)	$\dfrac{dL(r)}{dr} = 4\pi r^2\rho\epsilon$	Conservation of energy	$\epsilon = 0$
(e)	$P = \dfrac{\rho kT}{\mu m_h}$	Equation of state	$P = P(\rho)$
(f)	$\dfrac{dT}{dr} = f(P, T, \rho)$		The equation of state does not depend on T
(g)	$\kappa = \kappa(P, T, \rho)$		κ is irrelevant if $\epsilon = 0$

(6.3.1)

For relativistic stellar models, we need only solve equations (6.3.1a) through (6.3.1c) and (6.3.1e) subject to certain boundary conditions. Combining equations (6.3.1b) and (6.3.1c), we have just three equations in three unknowns – M(r), P, and ρ. Two of the equations are first-order differential equation requiring two constants of integration. One additional eigenvalue of the problem is required because we must specify the type (mass) of star we wish to make.

Thus,

$$P(R) = 0 \qquad M(R) = M \qquad M(0) = 0 \tag{6.3.2}$$

For the eigenvalue, we might just as well have specified the central pressure for that would lead to a specific star and would make the problem an initial value problem. We can gain some insight into the effects of general relativity by looking at a concrete example.

b A Simple Model

The reduction of the equation of state to the form $P = P(\rho)$ is reminiscent of the polytropic equation of state. For polytropes, the combination of the equation of state with hydrostatic equilibrium led to the Lane-Emden equation which specified the entire structure of the star subject to certain reasonable boundary conditions. To be sure, we could write a similar "relativistic" Lane-Emden equation for relativistic polytropes, but instead we take a different approach. Let us consider a situation where the constraint presented by the equation of state is replaced by a direct constraint on the density. While this does not result in a polytropic equation of state, it is illustrative and analytic, allowing for the solution to be obtained in closed form. Assume the density to be constant, so that

$$\rho(r) = \rho_0 = \text{constant} \tag{6.3.3}$$

The first of the two remaining equations of stellar structure then has the direct solution

$$M(r) = \frac{4\pi r^3 \rho_0}{3} \tag{6.3.4}$$

while the Oppenheimer-Volkoff equation of hydrostatic equilibrium becomes

$$\frac{dP(r)}{dr} = -\frac{4\pi G r \rho_0^2 [1 + P/(\rho_0 c^2)][1 + 3P/(\rho_0 c^2)]}{3[1 - 8\pi G \rho_0 r^2/(3c^2)]} \tag{6.3.5}$$

This equation has an analytic solution which can be obtained by direct, albeit somewhat messy, integration. We can facilitate the integration by introducing the variables

$$y = \frac{P}{\rho_0} \qquad \gamma = \frac{8\pi G\rho_0}{3c^2} = \frac{2GM}{R^3c^2} \tag{6.3.6}$$

and rewrite the equation for hydrostatic equilibrium as

$$\frac{dy}{dr} = -\frac{1}{2}\gamma c^2 \frac{(1 + y/c^2)(1 + 3y/c^2)r}{1 - \gamma r^2} \tag{6.3.7}$$

which is subject to the boundary condition $y(\mathrm{R}) = 0$. With zero as a value for the surface pressure, the solution of equation (6.3.7) is

$$y = c^2 \frac{(1 - \gamma r^2)^{1/2} - (1 - \gamma R^2)^{1/2}}{3(1 - \gamma R^2)^{1/2} - (1 - \gamma r^2)^{1/2}} \tag{6.3.8}$$

in terms of physical variables this is

$$P(r) = \rho_0 c^2 \frac{[1 - 2GMr^2/(R^3c^2)]^{1/2} - [1 - 2GM/(Rc^2)]^{1/2}}{3[1 - 2GM/(Rc^2)]^{1/2} - [1 - 2GMr^2/(R^3c^2)]^{1/2}} \tag{6.3.9}$$

Now we evaluate equation (6.3.9) for the central pressure by letting r go to zero. Then

$$P_c = \rho_0 c^2 \frac{1 - [1 - 2GM/(Rc^2)]^{1/2}}{3[1 - 2GM/(Rc^2)]^{1/2} - 1} \tag{6.3.10}$$

As the central pressure rises, the star will shrink, reflecting the larger effects of gravity so that

$$\lim_{P_c \to \infty} R = \frac{9}{8}\frac{2GM}{c^2} = \frac{9}{8}R_s \tag{6.3.11}$$

where R_s is the Schwarzschild radius. This implies that the smallest stable radius for such an object would be slightly larger than its Schwarzschild radius. A more reasonable limit on the central pressure would be to limit the speed of sound to be less than or equal to the speed of light. A sound speed in excess of the speed of light would suggest conditions where the gas would violate the principle of causality. Namely, sound waves could propagate signals faster than the velocity of light. Since P/ρ_0 is the square of the local sound speed, consider

$$\lim_{P_c \to c^2\rho_0} R = \tfrac{4}{3}R_s \tag{6.3.12}$$

This lower value for the central pressure yields a somewhat larger minimum radius. Since any reasonable equation of state will require that the density monotonically decrease outward and since causality will always dictate that the sound speed be less than the speed of light, we conclude that any stable star must have a radius R such that

$$R \geq \tfrac{4}{3} R_s \tag{6.3.13}$$

In reality, this is an extreme lower limit, and neutron stars tend to be rather larger and of the order of 4 or 5 Schwarzschild radii. Nevertheless, neutron stars still represent stellar configurations in which the general theory of relativity plays a dominant role.

c Neutron Star Structure

The larger size of actual neutron stars, compared to the above limit, results from detailed consideration of the physics that specifies the actual equation of state. Although this is still an active area of research and is likely to be so for some time, we will consider the results of an early equation of state given by Salpeter[5,6]. He shows that we can write a parametric equation of state in the following way:

$$\begin{aligned} P &= \tfrac{1}{3}K\left(\operatorname{Sinh} t - 8\operatorname{Sinh}\frac{t}{2} + 3t\right) \\ \rho &= K(\operatorname{Sinh} t - t) \end{aligned} \tag{6.3.14}$$

where

$$\begin{aligned} K &= \frac{\pi\mu_0^4 c^5}{4h^3} \\ t &= 4\operatorname{Log}\left\{\frac{\hat{p}}{\mu_0 c} + \left[1 + \left(\frac{\hat{p}}{\mu_0 c}\right)^2\right]^{1/2}\right\} \end{aligned} \tag{6.3.15}$$

and $\hat{\rho}$ is the maximum Fermi momentum and may depend weakly on the temperature. The relationship between the mass and central density is shown in Figure 6.1. If one includes the energy losses from neutrinos due to inverse beta decay, there exists a local maximum for the mass at around 1 solar mass.

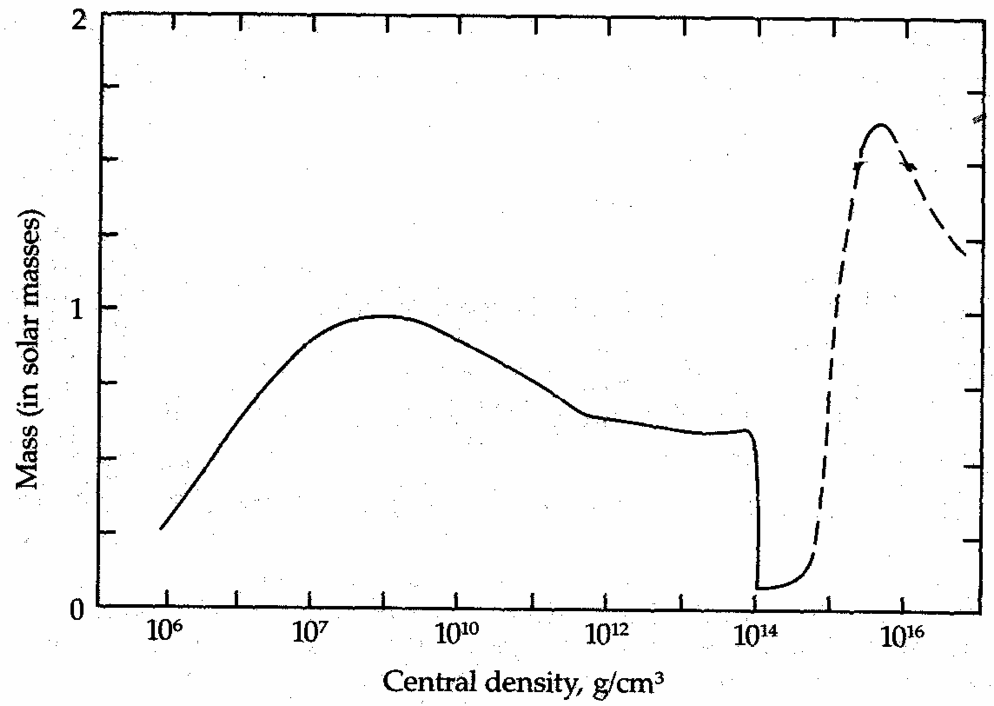

Figure 6.1 shows the variation of the mass of a degenerate object central density. The large drop in the stable mass at a density of about 10^{14} gm/cm^3 represents the transition from the electron degenerate equation of state to the neutron degenerate equation of state.

More recent modifications to the equation of state show a second maximum occurring at slightly more than 2 solar masses. Considerations of causality set an absolute upper limit for neutron stars at about $5M_{\odot}$. So there exists a mass limit for neutron stars, as there does for white dwarfs, and it is probably about $2.5M_{\odot}$. However, unlike the Chandrasekhar limit, this mass limit arises because of the effects of the general theory of relativity. As we shall see in the next section, this is also true for the mass limit of white dwarfs.

We have not said anything about the formidable problems posed by the formulation of an equation of state for material that is unavailable for experiment. To provide some insight into the types of complications presented by the equation of state, we show below, in Figure 6.2 the structure of a neutron star as deduced by Rudermann[7].

The equation of state for the central regions of such a star still remains in doubt as the Fermi energy reaches the level for the formation of hyperons. Some people have speculated that one might reach densities sufficient to yield a "quark soup". Whatever the details of the equation of state, they matter less and less as one approaches the critical mass. Gravity begins to snuff out the importance of the local microphysics. By the time one reaches a configuration that has contracted within its Schwarzschild radius only the macroscopic properties of total mass, angular momentum, and charge can be detected by an outside observer (for more on this subject see Olive, 1991).

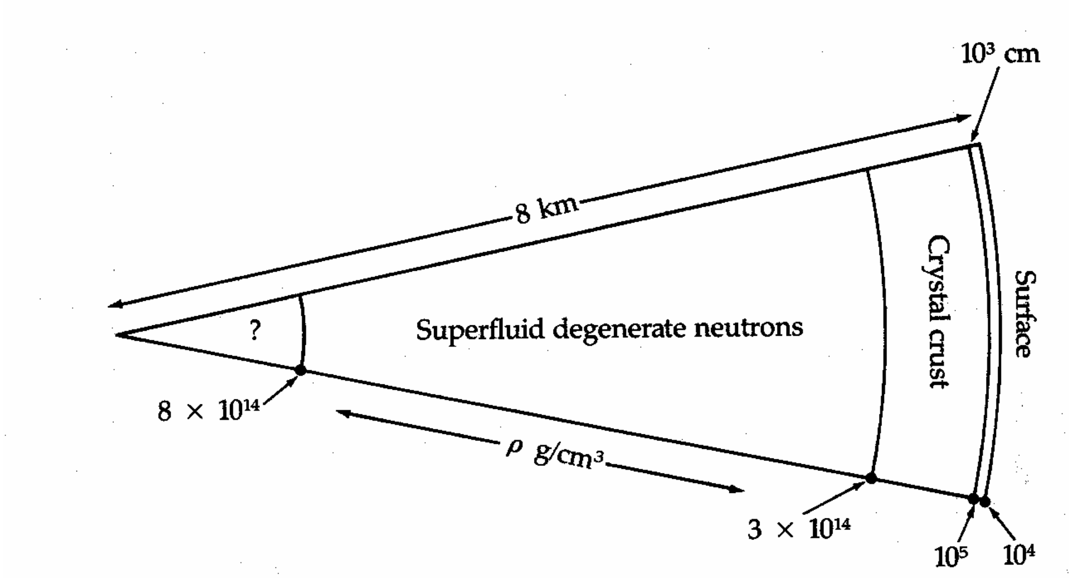

Figure 6.2 shows a section of the internal structure of a neutron star. The formation of crystal structure in the outer layers of the neutron star greatly complicates its equation of state. Its structure may be testable by observing the shape changes of rapidly rotating pulsars as revealed by discontinuous changes in their spin rates as they slow down.

Although this ultimate result occurs only when the object has reached the Schwarzschild radius, aspects of its approach are manifest in the insensitivity of the global structure to the equation of state as the limiting radius is approached. This has the happy result for astronomy that the mass limit for neutron stars can comfortably be set at around 2½ $M_\odot$ regardless of the vagaries of the equation of state. It has an unhappy consequence for physics in that neutron stars will prove a difficult laboratory for testing the details of the equation of state for high- density matter.

6.4 Relativistic Polytrope of Index 3

In Chapter 2, we remarked that the equation of state for a totally relativistic degenerate gas was a polytrope of index 3. In addition, we noted that an object dominated by radiation pressure would also be a polytrope of index $n = 3$. In the first category we find the extreme white dwarfs, those nearing the Chandrasekhar degeneracy limit. In the second category we find stars of very great mass where, from the β^* theorem, we can expect the total pressure to be very nearly that of the pressure from photons. It is somewhat curious that such different types of stars should have their structures given by the same equilibrium model. However, both types are dominated by relativistic (in the sense of the special theory of relativity) particles, and this aspect of the gas is characterized by a polytrope of index $n = 3$.

Our approach to the study of these objects will be a little different from our previous discussions of stellar structure. Rather than concentrate on the internal properties and physics of these objects, we consider only their global properties, such as mass, radius, and internal energy. This will be sufficient to understand their stability and evolutionary history. The ideal vehicle for such an investigation is the Virial theorem.

a Virial Theorem for Relativistic Stars

The Virial theorem for relativistic particles differs somewhat from that derived in Chapter 1. The effect of special relativity on the "mass" or momentum of such particles increases the gravitational energy required to confine the particles as the internal energy increases (see Collins[8]). Thus, for stable configurations, instead of

$$2\mathbf{T} + \mathbf{\Omega} = 0 \tag{6.4.1}$$

we get

$$\mathbf{T} + \mathbf{\Omega} = 0 \tag{6.4.2}$$

which specifies the total energy of the configuration as

$$E = \mathbf{T} + \mathbf{\Omega} = 0 \tag{6.4.3}$$

This is sometimes called the binding energy because it is the energy required to disperse the configuration throughout space. Thus polytropes of index $n = 3$ are neutrally stable since it would take no work at all to disperse them and as such these polytropes represent a limiting condition that can never be reached. To investigate the fate of objects approaching such a limit, it is necessary to look at the behavior of those conditions that lead to small departures from the limit. One of those conditions is the distortion of the metric of space caused by the matter-energy itself and so well described by the general theory of relativity.

Phenomenologically, we may view the effects of general relativity as increasing the effective "force of gravity". Thus, as we approach the limiting state of the relativistic polytrope, we would expect the effects of general relativity to cause the configuration to become unstable to collapse. So it is general relativity which sets the limit for the masses of white dwarfs, not the Pauli Exclusion Principle, just as general relativity set the limit for the masses of neutron stars. We could also expect such an effect for super-massive stars dominated by photon pressure.

To quantify these effects, we shall have to appeal to the Virial theorem in a non-Euclidean metric. Rather than re-derive the Boltzmann transport equation for the Schwarzschild metric, we obtain the relativistic Euler-Lagrange equations of hydrodynamic flow and take the appropriate spatial moments. We skip directly to the result of Fricke[9].

$$\frac{1}{2}\frac{d^2 I_r}{dt^2} = 3\int_V P\,dV + \Omega + \frac{G}{c^2}\left[\int_V \frac{M(r)(P+\rho\dot{r}^2)}{r}\,dV - 3\int_V \frac{GM^2(r)\rho}{r^2}\,dV\right] \tag{6.4.4}$$

Here I_r is the moment of inertia defined about the center of the Schwarzschild metric. The effects of general relativity are largely contained in the third term in brackets which is multiplied by G/c^2 and contains the additions to the potential energy of the kinetic energy of the gas particles (as represented by the pressure) and the kinetic energy of mass motions of the configuration (as represented by $\dot{r}^2\rho$). The physical interpretation of the second term in the brackets is more obscure. For want of a better description, it can be viewed as a self-interaction term arising from the nonlinear nature of the general theory of relativity. Except for the relativistic term, equation (6.4.4) is very similar to its Newtonian counterpart in Chapter 1 [equation (1.2.34)]. The effect of the internal energy is included in the term $3\int_V PdV$. Since we will be considering stars that are near equilibrium, we take the total kinetic energy of mass motions to be zero. The $\dot{r}^2\rho$ term was included in the relativistic term to emphasize its relativistic role.

A common technique in stellar astrophysics is to perform a variational analysis of the Virial theorem as expressed by equation (6.4.4), but such a process is quite lengthy. Instead, we estimate the effects of general relativity by determining the magnitude of the relativistic terms as $\gamma \rightarrow (4/3)$. Obviously if the left hand side of equation (6.4.4) becomes negative, the star will begin to acceleratively contract and will become unstable. Thus we investigate the conditions where the star is just in equilibrium. Replacing $3\int_V PdV$ with its equivalent in terms of the internal energy [see equation (5.4.2)], we get

$$3(\gamma - 1)U + \Omega = -\frac{G}{c^2}\int_V \frac{M(r)P}{r}\,dV + \frac{3G^2}{c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV \tag{6.4.5}$$

Now, since $\gamma = 4/3$ is a limiting condition, let

$$\gamma = \tfrac{4}{3} + \epsilon \qquad n = 3(1 - 3\epsilon) \tag{6.4.6}$$

where ε is positive. We may use the variational relation between the internal and potential energies

$$\delta U = -3(\gamma - 1)^{-1}\,\delta\Omega \tag{6.4.7}$$

(see Chandrasekhar[10]), and we get

$$\lim_{\gamma \to 4/3+\epsilon} [3(\gamma - 1)U + \Omega] = (1 + 3\epsilon)(U_0 + \delta U) + \Omega_0 + \delta\Omega = -3\Omega_0\epsilon \tag{6.4.8}$$

Here the subscript $_0$ denote the value of quantities when $\gamma = 4/3$, and, $U_0 = -\Omega_0$ for that value of g so the Virial theorem becomes

$$3\epsilon\Omega_0 = \frac{G}{c^2}\int_V \frac{M(r)P}{r}\,dV - \frac{3G^2}{c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV \tag{6.4.9}$$

We may now estimate the magnitude of the relativistic terms on the right-hand side as follows. Consider the first term where

$$\frac{G}{c^2}\int_V \frac{M(r)P}{r}\,dV = \frac{G}{c^2}\overline{\left(\frac{M}{r}\right)}\left(\frac{-\Omega_0}{3}\right) \approx \frac{GM}{Rc^2}\left(\frac{-\Omega_0}{3}\right) = -\frac{1}{6}\left(\frac{R_s}{R}\right)\Omega_0 \tag{6.4.10}$$

Here we have taken the pressure weighted mean of (M/R) to be M/R, and R_s is the Schwarzschild radius for the star. The second term can be dealt with in a similar manner, so

$$\frac{3G^2}{c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV = \frac{3G^2}{c^2}\overline{\left(\frac{M}{r}\right)^2} M \approx \frac{3GM^2}{2R}\frac{R_s}{R} = -\frac{R_s}{R}\Omega_0 \tag{6.4.11}$$

Again, we have replaced the mean of M/R by M/R. Since the means of the two terms are not of precisely the same form, we expect this approach to yield only approximate results. Indeed, the central concentration of the polytrope will ensure that both terms are underestimates of the relativistic effects. Even worse, the mean-square of M(r)/r in equation (6.4.11) will yield an even larger error than that of equation (6.4.10). Since the terms differ in sign, the combined effect could be quite large. However, we may be sure that the result will be a lower limit of the effects of general relativity, and the approximations do demonstrate the physical nature of the terms. With this large caveat, we shall proceed. Substituting into equation (6.4.9), we get

$$\frac{R}{R_s} \approx \frac{5}{18\epsilon} \tag{6.4.12}$$

Now all that remains to be done is to investigate how γ → 4/3 in terms of the defining parameters of the star (M, L, R), and we will be able to estimate when the effects of general relativity become important.

b Minimum Radius for White Dwarfs

We have indicated that the effects of general relativity should bring about the collapse of a white dwarf as it approaches the Chandrasekhar limiting mass. If we can characterize the approach of g to 4/3 in terms of the properties of the star, we will know how close to the limiting mass this occurs. As γ → 4/3, the degeneracy parameter in the parametric degenerate equation of state approaches infinity. Carefully expanding f(x) of equation (1.3.14) and determining its behavior as x → 4 we get

$$\lim_{x\to\infty} f(x) = 2(x^4 - x^2) \tag{6.4.13}$$

From the polytropic equation of state

$$\gamma = \frac{d(\ln P)}{d(\ln \rho)} \tag{6.4.14}$$

Evaluating the right-hand side from the parametric equation of state [equation (1.3.14)] and the result for f(x) given by equation (6.4.13), we can combine with the definition of e from equation (6.4.6) to get

$$\epsilon \approx \frac{2x^{-2}}{3} \tag{6.4.15}$$

If we neglect the effects of inverse beta decay in removing electrons from the gas, we can write the density in terms of the electron density and, with the aid of equation (1.3.14), in terms of the degeneracy parameter x.

$$\rho = \frac{8\pi c^3 m_e^3 \mu_e m_h x^3}{3h^3} \tag{6.4.16}$$

If we approximate the density by its mean value, we can solve for the average square degeneracy parameter for which we can expect collapse.

$$\bar{x}^2 = \left(\frac{9h^3}{32\pi^2 c^3 m_e^3 \mu_e m_h}\right)^{2/3}\left(\frac{M}{R^3}\right)^{2/3} = (7.12\times 10^6)\mu_e{}^{2/3}\left(\frac{M_\odot}{M}\right)^{4/3}\left(\frac{R_s}{R}\right)^2 \quad (6.4.17)$$

Combining this with equations (6.4.15) and (6.4.12), we obtain an estimate for the manner in which the minimum stable radius of a white dwarf depends on mass as the limiting mass is approached.

$$\frac{R}{R_s} \approx 144\mu_e^{-2/9}\left(\frac{M_\odot}{M}\right)^{4/9} \quad (6.4.18)$$

A more precise calculation involving a proper evaluation of the relativistic integrals and evaluation of the average internal degeneracy by Chandrasekhar and Trooper[11] yields a value of 246 Schwarzschild radii for the minimum radius of a white dwarf, instead of about 100 given by equation (6.4.18). We can then combine this with the mass-radius relation for white dwarfs to find the actual value of the mass for which the star will become unstable to general relativity. This is about 98 percent of the value given by the Chandrasekhar limit, so that for all practical purposes the degeneracy limit gives the appropriate value for the maximum mass of a white dwarf.

However, massive white dwarfs do not exist because general relativity brings about their collapse as the star approaches the Chandrasekhar limit. This point is far more dramatic in the case of neutron stars. Here the general relativistic terms bring about collapse long before the entire star becomes relativistically degenerate. A relativistically degenerate neutron gas has much more kinetic energy per gram than a relativistically degenerate electron gas, since a relativistic particle must have a kinetic energy greater than its rest energy, by definition. To contain such a gas, the gravitational forces must be correspondingly larger, which implies a greater importance for general relativity. Indeed, to confine a fully relativistically degenerate configuration, it would be necessary to restrict it to a volume essentially bounded by its Schwarzschild radius. This is not to say that the cores of neutron stars cannot be relativistically degenerate. Indeed they can, but the core is contained by the weight of the nonrelativistically degenerate layers above as well as its own self-gravity.

c Minimum Radius for Super-massive Stars

Since the early 1960s, super-massive stars have piqued the interest of some. It was thought that such objects might provide the power source for quasars. While their existence might be ephemeral, if super-massive stars were formed in sufficient numbers, their great luminosity might provide a solution to one of the foremost problems of the second half of the twentieth century. However, truly massive stars are subject to the same sort of instability as we investigated for white dwarfs. Indeed, the problem was first discussed by W. Fowler[12,13] for the super-massive stars and later extended by Chandrasekhar and Trooper[11] to white dwarfs. Finally the problem was re-discussed by Fricke[9] and the effect of the metric on the relativistic integrals was correctly included.

By now you should not be surprised that such an instability exists because we know that massive stars are dominated by radiation pressure and can be well represented by polytropes of index $n = 3$. For super-massive stars, the departure from being a perfect relativistic polytrope results from some of the total energy being provided by the kinetic energy of the gas particles. To quantify the instability, we may proceed as we did with the white dwarf analysis. If we write the equilibrium Virial theorem and split the $3\int_V P dV$ term into a sum of the gas pressure and the radiation pressure, then we get

$$0 = 3\int_V \beta P\,dV + 3\int_V (1-\beta)P\,dV + \Omega + \frac{G}{c^2}\int_V \frac{M(r)P}{r}\,dV - \frac{3G^2}{c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV \tag{6.4.19}$$

However, the total energy of the configuration is

$$E = \frac{3}{2}\int_V \beta P\,dV + 3\int_V (1-\beta)P\,dV + \Omega + \frac{3G}{c^2}\int_V \frac{M(r)P}{r}\,dV - \frac{3G^2}{2c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV \tag{6.4.20}$$

Subtracting equation (6.4.19) from (6.4.20), we get

$$E = -\frac{3}{2}\int_V \beta P\,dV + \frac{2G}{c^2}\int_V \frac{M(r)P}{r}\,dV + \frac{3G^2}{2c^2}\int_V \frac{M^2(r)\rho}{r^2}\,dV \approx 0 \tag{6.4.21}$$

When the total energy of the configuration becomes zero, we will have reached its minimum stable radius. Making the same approximations for the relativistic integrals that were made for the white dwarf analysis, we get

$$\frac{R_0}{R_s} \approx \frac{5}{3\beta} \tag{6.4.22}$$

From equation (2.2.11) we saw that the central value of beta, β_c, was bounded by the mass, so that

$$\beta_c \propto M^{-1/2} \tag{6.4.23}$$

Using the constant of proportionality implied by equation (2.2.11) and combining with equation (6.4.22), we get

$$\frac{R_0}{R_s} \simeq 0.18\left(\frac{M}{M_\odot}\right)^{1/2} \tag{6.4.24}$$

Thus a super-massive star of $10^8 M_\odot$ will become unstable at about 1800 Schwarzschild radii or about 14 AU. In units of the Schwarzschild radius, this result is rather larger than that for white dwarfs. This can be qualitatively understood by considering the nature of the relativistic particles providing the majority of the internal pressure in the two cases. The energy of the typical photon providing the radiation pressure for a super-massive star is far less than the energy of a typical degenerate electron whose degenerate pressure provides the support in a white dwarf. Thus a weaker gravitational field will be required to confine the photons as compared to the electron. This implies that as the total energy approaches zero, the mass required to confine the photons can be spread out over a larger volume, when measured in units of the Schwarzschild radius, than is the case for the electrons. This argument implies that neutron stars should be much closer to their Schwarzschild radius in size, which is indeed the case.

Perhaps the most surprising aspect of both these analyses is that general relativity can make a significant difference for structures that are many hundreds of times the dimensions that we usually associate with general relativity.

6.5 Fate of Super-massive Stars

The relativistic polytrope can be used to set minimum sizes for both white dwarfs and very massive stars. However, super-massive stars are steady-state structures and will evolve, while white dwarfs are equilibrium structures and will remain stable unless they are changed by outside sources. Let us now see what can be said about the evolution of the super-massive stars.

a Eddington Luminosity

Sir Arthur Stanley Eddington observed that radiation and gravitation both obey inverse-square laws and so there would be instances when the two forces could be in balance irrespective of distance. Thus there should exist a maximum luminosity for a star of a given mass, where the force of radiation on the surface material would exactly balance the force of gravity. If we balance the gravitational acceleration against the radiative pressure gradient [equation (4.2.11)] for electron scattering, we can write

$$\nabla P_r = -\frac{\sigma_e \rho L(r)}{4\pi c R^2} = -\frac{GM\rho}{R^2} \tag{6.5.1}$$

Therefore, any object that has a luminosity greater than

$$L_{\text{Edd}} = \frac{4\pi Gc}{\sigma_e} M \approx (1.3 \times 10^{38})\left(\frac{M}{M_\odot}\right) \quad \text{erg/s} \tag{6.5.2}$$

will be forced into instability by its own radiation pressure. This effectively provides a mass-luminosity relationship for super-massive stars since these radiation-dominated configurations will radiate near their limit.

b Equilibrium Mass-Radius Relation

If we now assume that the star can reach a steady-state, that represents a near-equilibrium state on a dynamical time, then the energy production must equal the energy lost through the luminosity. Eugene Capriotti[14] has evaluated the luminosity integral and gets

$$L = \int_V \rho\epsilon\, dV \simeq (2.9 \times 10^{-67})\left(\frac{M^{8.625}}{R^{16.25}}\right) \quad \text{erg/s} \tag{6.5.3}$$

We can assume that massive stars will derive the nuclear energy needed to maintain their equilibrium from the CNO cycle, can evaluate e as indicated in Chapter 3 [equation (3.3.19)], and can evaluate the central term of equation (6.5.3) to obtain the approximate relation on the right. Assuming that the stars will indeed radiate at the Eddington luminosity, we can use equation (6.5.2) to find

$$R_e \approx (1.7 \times 10^{11})\left(\frac{M}{M_\odot}\right)^{0.47} \quad \text{cm} \tag{6.5.4}$$

Thus we have a relation between the mass and radius for any super-massive star that would reach equilibrium through the production of nuclear energy. However, we have yet to show that the star can reach that equilibrium state.

c Limiting Masses for Super-massive Stars

Let us add equations (6.4.19) and (6.4.20) taking care to express the relativistic integrals as dimensionless integrals by making use of the homology relations for pressure and density, and get for the total energy:

$$E = -\frac{1}{2}\bar{\beta}\Omega + \frac{2G^2M^3}{R^2c^2}\int_0^1 \left[\frac{M(r)R}{Mr}\right]\frac{P}{P_c}\frac{\rho_c}{\rho}\frac{dM(r)}{M} - \frac{9G^2M^3}{2R^2c^2}\int_0^1 \left[\frac{M(r)R}{Mr}\right]^2 \frac{dM(r)}{M} \tag{6.5.5}$$

We must be very careful in evaluating these integrals, for any polytrope in Euclidean space as the radial coordinate used to obtain those integrals is defined by the Schwarzschild metric (see Fricke[9] p. 942). We must do so here, since we will not be content to find a crude result for the mass limits.

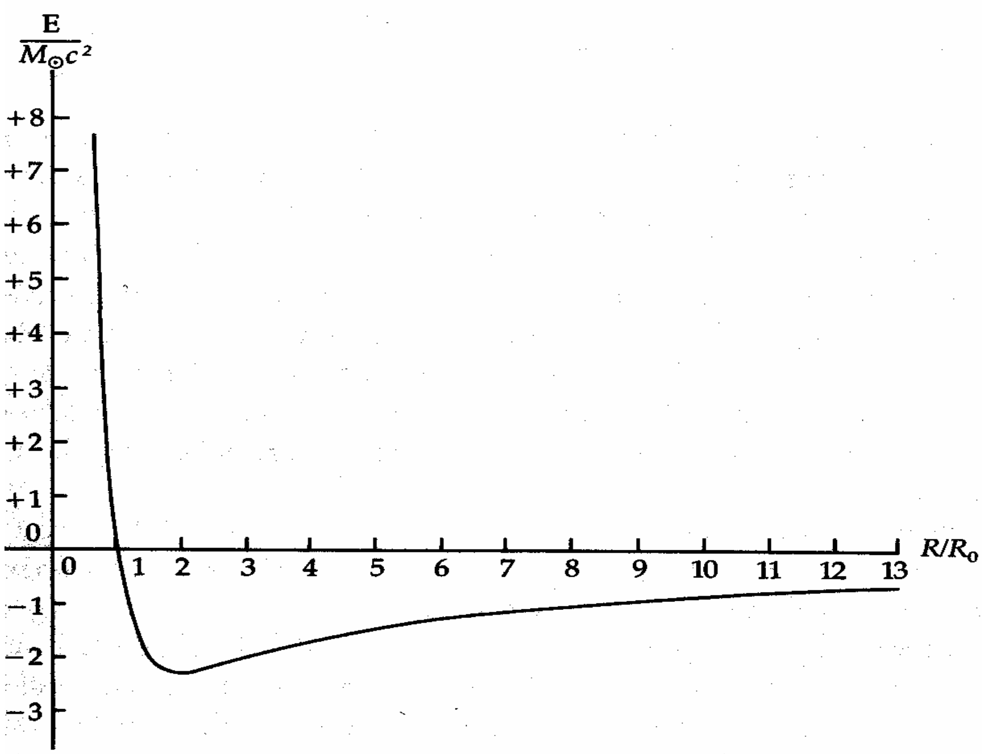

Figure 6.3 shows the variation of the binding energy in units of the rest energy of the sun as a function of the radius in units of the minimum stable radius. In is clear that a minimum (most negative) binding energy exists and that the minimum is a specific value for all super massive stars.

Replacing β by its limiting value given by the β* theorem and evaluating the relativistic integrals for a polytrope of index $n = 3$, we obtain

$$E = -\frac{27GM_\odot^2}{4R_\odot}\left(\frac{M}{M_\odot}\right)^{3/2}\frac{R_\odot}{R} + 5.07\frac{G^2M_\odot^3}{R_\odot^2c^2}\left[\left(\frac{M}{M_\odot}\right)^{3/2}\frac{R_\odot}{R}\right]^2 \tag{6.5.6}$$

If we now seek the radial value for which E = 0, we get

$$R_0 \approx 0.756\left(\frac{GM_\odot}{c^2}\right)\left(\frac{M}{M_\odot}\right)^{3/2} = (1.1 \times 10^5)\left(\frac{M}{M_\odot}\right)^{3/2} \quad \text{cm}$$
$$= 0.37R_s\left(\frac{M}{M_\odot}\right)^{1/2} \tag{6.5.7}$$

Comparing this result with equation (6.4.22), we see that our crude approximation of the relativistic integrals was low by about an order of magnitude. Equation (6.5.6) is quadratic in $M^{3/2}/R$ and will become positive at small R. Figure 6.3 shows the dependence of the binding energy on the radius. Differentiation of equation (6.5.6) shows that the greatest (most negative) binding energy will occur at

$$R_m = 2R_0 \tag{6.5.8}$$

and this corresponds to an energy of

$$E_m = -2.25M_\odot c^2 \tag{6.5.9}$$

This energy is a constant because of the quadratic nature of the energy equation. The relativistic terms simply vary as the next power of $[M^{3/2}/R]$ compared to the Newtonian terms. Hence the minimum will depend only on physical constants.

A star that is contracting toward its equilibrium position may reach equilibrium for any radial value that is greater than, or equal to R_m, providing an energy source exists to replace the energy lost to space. We have already found the equilibrium radius for energy produced by the CNO cycle [equation (6.5.4)]. Combining that with the minimum energy radius, we find

$$\frac{M}{M_\odot} \le 5.2 \times 10^5 \tag{6.5.10}$$

Thus any star with a mass less than about half a million solar masses can come to equilibrium burning hydrogen via the CNO cycle, albeit with a short lifetime. More massive stars are destined to continue to contract. Of course, more massive stars will produce nuclear energy at an ever-increasing rate as their central temperatures rise. However, the rate of energy production cannot increase without bound. This is suggested by the declining exponents of the temperature dependence shown in Table 3.4. The nuclear reactions that involve β decay set a limit on how fast the CNO cycle can run, and β decay is independent of temperature. So there is a maximum rate at which energy can be produced by the CNO cycle operating in these stars.

If the nuclear energy produced is sufficient to bring the total energy above the binding energy curve, the star will explode. However, should the energy not be produced at a rate sufficient to catch the binding energy that is rising due to the relativistic collapse, the star will continue an unrestrained collapse to the Schwarzschild radius and become a black hole. Which scenario is played out will depend on the star's mass. For these stars, the temperature gradient will be above the adiabatic gradient, so convection will exist. However, the only energy transportable by convection is the kinetic energy of the gas, which is an insignificant fraction of the internal energy. Therefore, unlike normal main sequence stars, although it is present, convection will be a very inefficient vehicle for the transport of energy. This is why the star remains with a structure of a polytrope of index $n = 3$ in the presence of convection. The pressure support that determines the density distribution comes entirely from radiation and is not governed by the mode of energy transport. We saw a similar situation for degenerate white dwarfs. The equation of state indicated that their structure would be that of a polytrope of index $n= 1.5$ (for nonrelativistic degeneracy) and yet the star would be isothermal due to the long mean free path of the degenerate electrons. However, the structure is not that of an isothermal sphere since the pressure support came almost entirely from the degenerate electron gas and is largely independent of the energy and temperature distribution of the ions.

The star will radiate at the Eddington luminosity, and that will set the time scale for collapse. Remember that the total energy of these stars is small compared to the gravitational energy. So most of the energy derived from gravitational contraction must go into supporting the star, and very little is available to supply the Eddington luminosity. This can be seen from the relativistic Virial theorem [equation (6.4.2)], which indicates that any change in the gravitational energy is taken up by the kinetic energy. Relativistic particles (in this case, photons) are much more difficult to bind by gravitation than ordinary matter; thus little of the gravitational energy resulting from collapse will be available to let the star shine. The collapse will proceed very quickly on a time scale that is much nearer to the dynamical time scale than the Kelvin-Helmholtz time scale. The onset of nuclear reactions will slow the collapse, but will not stop it for the massive stars.

A dynamical analysis by Appenzeller and Fricke[15,16] (see also Fricke[9]) shows that stars more massive than about $7.5 \times 10^5 M_\odot$ will undergo collapse to a black hole. Here the collapse proceeds so quickly and the gravity is so powerful that the nuclear reactions, being limited by β decay at the resulting high temperatures, do not have the time to produce sufficient energy to arrest the collapse. For less massive stars, this is not the case. Stars in the narrow range of $5\times10^5 M_\odot$ # M # $7.5\times10^5 M_\odot$ will undergo explosive nuclear energy generation resulting in the probable destruction of the star.

Nothing has been said about the role of chemical composition in the evolution of these stars. Clearly, if there is no carbon present, the CNO cycle is not available for the stabilization of the star. Model calculations show that the triple-a process cannot stop the collapse. For stars with low metal abundance, only the proton-proton cycle is available as an energy source. This has the effect of lowering the value of the maximum stable mass. Surprisingly, there is no range at which an explosion occurs. If the star cannot stabilize before reaching R_m, it will continue in a state of unrestrained gravitational collapse to a black hole. Thus, it seems unlikely that stars more massive than about a half million solar masses could exist. In addition, it seems unlikely that black holes exist with masses greater than a few solar masses and less than half a million solar masses. If they do, they must form by accretion and not as a single entity.

Problems

1. Describe the physical conditions that correspond to polytropes of different indices, and discuss which stars meet these conditions.

2. What modifications must be made to the classical equation of hydrostatic equilibrium to obtain the Oppenheimer-Volkoff equation of hydrostatic equilibrium?

3. Find the mass-radius law for super-massive stars generating energy by means of the proton-proton cycle. Assume that the metal abundance is very small.

4. Determine the mass corresponding to a white dwarf at the limit of stability to general relativity.

5. Evaluate the relativistic integrals in equation (6.4.4) for a polytrope of index $n = 3$. Be careful for the Euclidean metric appropriate for the polytropic tables is not the same as the Schwarzschild metric of the equation (see Fricke[9] p. 941).

6. Use the results of Problem 5 to reevaluate the minimum radius for white dwarfs.

7. Assuming that a neutron star can be represented by a polytrope with $\gamma = 3/2$, find the minimum radius for a neutron star for which it is stable against general relativity. To what mass does this correspond?

References and Supplemental Reading

1. Landau, L.: *On the Theory of Stars,* Physik Zeitz. Sowjetunion 1, 1932, pp. 285 - 288.

2. Oppenheimer, J.R., and Snyder, H.: *On Continued Gravitational Contraction*, Phys. Rev. 56, 1939, pp. 455 - 459.

3. Oppenheimer, J.R., and Volkoff, G.: *On Massive Neutron Cores*, Phys. Rev. 55, 1939, pp. 374 - 381.

4. Misner, C.W., Throne, K.S., and Wheeler, J.A.: *Gravitation*, W.H. Freeman, New York, 1970

5. Salpeter, E.E.: *Matter at High Densities*, Ann. Rev. Phys. 11, 1960, pp. 393 - 417.

6. ______________ *Energy and Pressure of a Zero Temperature Plasma*, Ap.J. 134, 1961, pp. 683 - 698.

7. Rudermann, M.: *Pulsars: Structure and Dynamics*, Ann. Rev. Astron. and Astrophys. 10, 1972, pp. 427 - 476.

8. Collins, G.W.,II,: *The Virial Theorem in Stellar Astrophysics*, Pachart Pubishing House, 1978, pp. 34 -37.

9. Fricke, K.J.: *Dynamical Phases of Supermassive Stars*, Ap. J. 183, 1973, pp. 941 - 958.

10. Chandrasekhar S.: *An Introduction to the Study of Stellar Structure*, Dover, New York, 1957, p.53 eq. (102).

11. Chandrasekhar,S., and Trooper, R.F.: *The Dynamical Instability of the White-Dwarf Configuration approaching the Limiting Mass*, Ap. J. 139, 1964, pp.1396 -1397.

12. Fowler, W.A.: *Massive Stars, Relativistic Polytropes, and Gravitational Radiation*, Rev. Mod. Phys., 36, 1964, pp. 545 - 555, p.1104 Errata,

13. Fowler, W.A.: *The Stability of Supermassive Stars*, Ap.J. 144, 1966, pp.180 -200.

14. Capriotti, E.R. private communication, 1986.

15. Appenzeller, I., and Fricke, K.: *Hydrodynamic Model Calculations for Supermassive Stars I: The Collapse of a Nonrotating $0.75 \times 10^6 M_\odot$ Star*, Astr. and Ap. 18, 1972, pp. 10 - 15.

16. Appenzeller, I., and Fricke, K.: *Hydrodynamic Model Calculations for Supermassive Stars II: The Collapse and Explosion of a Nonrotating $5.2 \times 10^5 M_\odot$ Star*, Astr. and Ap. 21, 1972, pp. 285 - 290.

For a more detailed view of the internal structure of neutron stars, one should see:

Baym,G., Bethe, H., and Pethick,C.J.: *Neutron Star Matter*, Nuc. Phys. A 175, 1971, pp. 225 - 271.

No introduction to the structure of degenerate objects would be complete without a reading of

Hamada,T. and Salpeter,E.E.: *Models for Zero-Temperature Stars*, Ap.J. 134, 1961, pp. 683 - 698.

I am indebted to E. R. Capriotti for introducing me to the finer points of super-massive stars, and most of the material in Sections 6.4b, 6.5b, and 6.5c was developed directly from his notes of the subject. Those interested in the historical development of Super-massive stars should read:

Hoyle, F., and Fowler, W.A.: *On the Nature of Strong Radio Sources*, Mon. Not. R. astr. Soc. 125, 1963, pp. 169 - 176,

Faulkner, J., and Gribbin, J.R.: *Stability and Radial Vibrational Periods of the Hamada Salpeter White Dwarf Models*, Nature 218, 1966, pp.734 - 736.

While there are many other contributions to the subject that I have not included, these will acquaint the readers with the important topics and the flavor of the subject.

After the initial edition published by W.H. Freeman in 1989 there have been numerous additions to the literature in this area. One of the most notable dealing with the structure of Neutron Stars and the Quark-Hadron phase transition is:

Olive, K., 1991, Science, 251, pp. 1197-1198.