Part II

Stellar Atmospheres

9

The Flow of Radiation Through the Atmosphere

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In Part I, we discussed the internal structure and history of stars with little reference to the actual appearance of the stars themselves. Yet virtually all we know of stars rests on the information that we receive from their surfaces, and so we need to understand those processes that affect the light radiated into space. At several points in the evolution of stars, their evolution was determined by the efficiency with which radiation could be lost from their atmospheres. Thus, the structure of the atmosphere may be expected to play a role in the evolution of the star itself. In addition, for some stars, the region which we call the atmosphere represents a substantial fraction of the radial extent of the star, so that the surface boundary conditions on the equations of stellar structure are set by the atmospheric structure. When this fact is important, the very meaning of a stellar radius becomes intertwined with the details of the atmospheric structure. With the absence of a clearly defined radius, the notion of an effective temperature linked to the stellar luminosity and radius becomes meaningless. Thus, the physical situation near the surface of a star must be treated differently from that of the interior. This transition zone from the relatively simple physics of the stellar interior to the emptiness of interstellar space is commonly known as the *stellar atmosphere*.

There is a tendency to think of the difference between the interior and the atmosphere of a star as distinct, as it is with earth. To be sure, the relative extent of the solar atmosphere compared to the interior is similar to that of earth, but the similarity ends there. There is no sharp interface between stellar atmospheres and interiors as commonly exist with planets. There is no material phase change at the interface. Indeed, for stars, the distinction between atmospheres and interiors is denoted by the failure of certain assumptions used in the study of stellar interiors.

The solution of the problem posed by the surface layers of a star is similar to that for the interior. We have to describe the behavior of the state variables P, T, and ρ with position in the star. However, an additional problem is posed by the atmosphere. We have to describe the energy distribution of photons as they leave the star, for this specifies the appearance of the star which is the fundamental tie with observation. Only if this description of the stellar spectrum agrees with that which is observed can we say that we have provided a successful description of the star.

The approach to finding the structure of the atmosphere can be largely divided into two parts. First, one determines the flow of radiation through the atmosphere, given the structure of the atmosphere. Second, having determined the radiation field throughout the surface layers, one corrects the atmospheric structure so that energy is conserved at all levels of the atmosphere. Since most of the energy is carried by radiation, the second condition usually amounts to the imposition of radiative equilibrium throughout the atmosphere. One then uses the improved structure to correct the radiation field and repeats the process until a self-consistent model is found. To carry out this procedure it is necessary to make some assumptions about the conditions that prevail in this transition zone between the interior and the space surrounding the star.

9.1 Basic Assumptions for the Stellar Atmosphere

a Breakdown of Strict Thermodynamic Equilibrium

The description of the energy distribution of the photons in the stellar interior was made particularly simple by the assumption that all constituents of the gas that made up the star were in their most probable macrostate, resulting from random or uncorrelated collisions. That is, they were in thermodynamic equilibrium. All aspects of such a gas can then be characterized by a single parameter, the temperature, which specifies the mean energy of the gas. All other aspects of the distribution function of the gas particles are described by the equilibrium distribution function for the respective kinds of particles.

The validity of this assumption relied on the fact that various components of the gas would undergo randomizing collisions within a volume where the state variables (specifically the temperature) could be considered constant. Within the deep interior of a star, these conditions are met as well as anywhere in the universe. However, any configuration must have a boundary, and it is there that we should expect this assumption to fail. Such is the case for stars. However, the manner of that failure has a peculiar characteristic in that the particles that make up a star are of two distinctly different types. The photons that make up such an important component of the gas behave quite differently from the particles that have a material rest mass. These photons follow different quantum statistics so that their equilibrium distribution functions are different − Bose-Einstein for the photons and generally maxwellian for everything else. In addition, the mean free path between collisions for material particles is very much less than that for photons. Thus, we would expect that the photons would be the first species of particles to be affected by the presence of a boundary, and this is indeed the case. As one moves outward through a star, the presence of the surface begins to affect the state of the gas when photons first begin to escape directly into space and fail to interact any longer with the material particles of the gas. Since the probability that a specific photon will escape depends on the atomic physics of the opacity corresponding to the photon's energy, we should not expect all photons to escape with equal facility. Thus, the photon distribution will depart progressively from that of the Planck's law as one approaches the boundary and our notion of STE will have broken down.

The increase in the photon mean free path brought about by the decreasing density introduces another problem not unrelated to that posed by the boundary. The variation of the state variables over a "typical" photon mean free path will become a significant fraction of the value of the variables themselves. Thus, the radiation field at any point near the boundary will be made up of photons originating in rather different physical environments. Thus, the characteristics of the radiation field will no longer be determined by the local values of the state variables, but will depend on the structure solution of the entire atmosphere. This global aspect of the properties of the local radiation field completely changes the mathematical formalism that describes the flow of radiation from that used in the interior.

b Assumption of Local Thermodynamic Equilibrium

It is a happy consequence of the difference between photons and particles with material rest mass that the mean free path for photons is generally very much greater than that for other particles. Thus, while the photons may sense the boundary, there is a substantial region where the material particles do not. The material particles continue to undergo collisions with other material particles and photons, the majority of which still represent their thermodynamic equilibrium distribution. Thus, the material particles of the gas will continue to behave as if they were in thermodynamic equilibrium as one approaches the boundary. Certainly, the point will be reached when collisions between material particles and other constituents of the gas will become sufficiently infrequent that the nonequilibrium photons of the gas will force departures from the Maxwell-Boltzmann energy distribution expected for particles in thermodynamic equilibrium. But by this point in the atmosphere (in many stars), the majority of the photons will have escaped, much of the stellar spectrum will have been established, and the atmospheric structure below this point will be determined. Thus, the notion that the distribution function for the material particles remains that obtained from the local values of the state variables in thermodynamic equilibrium, while the photon distribution does not, is a useful notion. It is called *local thermodynamic equilibrium* (LTE) and it is one of the central assumptions for much of the remainder of this book. To understand the physical situation that prevails when LTE fails, one must first understand the solution to the problems for which LTE is valid.

The effect of the boundary upon particles that lie within a mean free path of the boundary extends to convective blobs. In the stellar interior, we were able to make do with the crude mixing-length theory because the differences between the adiabatic gradient and that predicted by the mixing-length theory were so small that large errors in this difference became rather small errors in the actual gradient. This was due to the large size of the mixing length, which implied great efficiency for convective transport. This will no longer be the case in the stellar atmosphere, for it is not possible to have a mixing length greater than the local distance to the boundary, and that is the order of a photon mean free path. Thus, convection, should it even occur in the deeper sections of the atmosphere, will be nowhere as efficient as it was in the interior. The mixing-length theory, while crude, can be used to estimate the impact of convection on the atmospheric structure. Fortunately, radiation dominates, by definition, in the outer sections of the atmosphere, and so convection will not be a major concern.

c Continuum and Spectral Lines

In describing the spectral energy distribution of the photons emerging from a star, it is traditional to distinguish between the smooth distribution of photons and the dark interruptions, or lack of photons, called *spectral absorption lines*. These features arise because the opacity of atomic bound-bound transitions is so large compared to that of bound-free and free-free processes that photons with energies corresponding to those bound-bound transitions do not sense the boundary until they are relatively near it. At this point in the atmosphere, the temperature has declined to the point where the emitted radiation is less intense than that originating deeper in the atmosphere. Thus, there will be fewer photons at the frequencies corresponding to the bound-bound transitions, giving rise to the absorption lines of stellar spectra.

Remember that the distinction between continuum and line is largely artificial, and often the continuum is shot through with myriads of weak lines. The utility of the concept persists, and we are careful to explain exactly what is meant by the distinction. Since a large section of this book is be devoted to the processes that give rise to spectral lines (and throughout that section we assume that the structure of the atmosphere is known), we assume that continuum processes and photons involved in those processes are the photons that determine the structure of the atmosphere.

d Additional Assumptions of Normal Stellar Atmospheres

Although some of the development of the theory of stellar atmospheres is presented in great generality, the basic focus of this book is on the theory of "normal" stars. This development is appropriate for most of the stars on the main sequence and some others. We indicate where the assumptions fail in the description of the atmospheres of other stars and what can be done about them, but for now we adopt the traditional assumptions of stellar atmospheres.

In addition to the assumption of LTE, we assume that the thickness of the atmosphere is small compared to the radius of the star. Under these conditions, the surface geometry may be assumed to be that of a plane- parallel slab of infinite thickness possessing a surface extending to infinity in all directions (see Figure 9.1). Since most of the stellar mass will reside inside the atmosphere, it is consistent with the plane-parallel atmosphere approximation to assume that the surface gravity is constant. Thus, the notion of hydrostatic equilibrium given in equation (2.1.6) simplifies to

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho}{r^2} = -g\rho \tag{9.1.1}$$

Furthermore, since no sources of energy are likely to be present in the stellar atmosphere and we need not worry about time dependent entropy terms, the conservation of energy [equation (7.1.1)] becomes

$$\nabla \cdot \vec{F} = 0 = \frac{dF}{dx} \qquad F = \text{const} = \sigma T_e^4 \tag{9.1.2}$$

If all the energy is to be carried by radiation, equation (9.1.2) ensures that the radiant flux in the atmosphere will be constant.

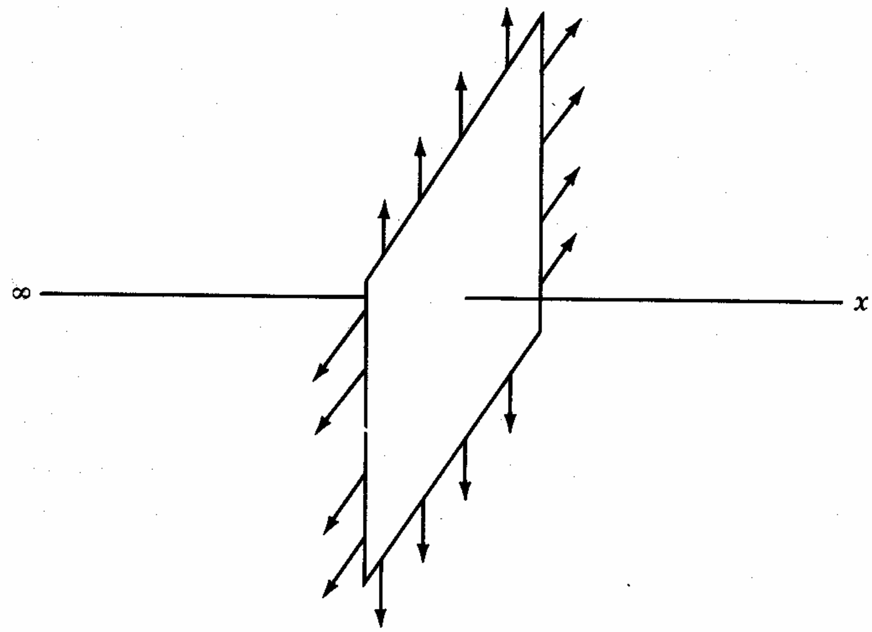

Figure 9.1 shows the semi-infinite plane that is appropriate for describing the local conditions for stars with thin atmospheres.

Thus these are the fundamental assumptions for the theory of normal stellar atmospheres:

1. LTE prevails. All properties of the material gas can be specified in terms of the local thermodynamic variables.
2. The atmospheric structure is affected by the continuum opacity only.
3. The local geometry is that of a plane-parallel slab.
4. The local surface gravity can be regarded as constant throughout the atmosphere.
5. All energy is carried by radiation, and there are no sources of energy within the atmosphere.

Under these conditions, in addition to the chemical composition, only two parameters are required to specify the structure of the atmosphere: they are

$$g = \frac{GM(r)}{R^2} \qquad T_e^4 = \frac{L}{4\pi\sigma R^2} \tag{9.1.3}$$

Since R^{-2} appears in both the expressions for T_e and g; it is no longer an independent parameter required for specifying the atmospheric structure. This is a result of the plane-parallel approximation and does not represent a fundamental difference between the theory of stellar atmospheres and the theory of stellar interiors. If that approximation were to be relaxed, R would be required, indicating that the same parameters (M, L, and R) are necessary for the specification of the model's structure as were required for stellar interiors.

9.2 Equation of Radiative Transfer

In this section we describe, with some generality, the flow of radiation through the outer layers of the star. We developed the formalism for this in Chapter 1 in the form of the Boltzmann transport equation. This formalism basically allows us to describe the flow of any ensemble of particles from one point to another as long as we include all mechanisms for the "creation" and "destruction" of those particles in phase space. In Chapter 1, we used the Boltzmann transport equation to describe the flow of material particles and their momentum through an arbitrary medium. Now we consider the analogous flow of photons.

For material particles, three of the phase space coordinates were velocity. But such coordinates are clearly inappropriate for photons, so we replaced those coordinates with the three components of the photon momentum. This enabled us to write equation (1.2.5) in momentum coordinates so that

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i}\right) = S \tag{9.2.1}$$

For describing the flow of photons, f represents the density in phase space of photons while S describes their creation and destruction at a local point in phase space. However, it is traditional to describe the photon phase density in terms of a quantity called the specific intensity.

a Specific Intensity and Its Relation to the Density of Photons in Phase Space

The specific intensity is an energy-like quantity that describes the flow of energy in a particular direction, through a differential area, into a differential solid angle, per unit frequency and time (see Figure 9.2). Remember that the momentum of a photon is just its energy divided by the speed of light:

$$p = \frac{h\nu}{c} \tag{9.2.2}$$

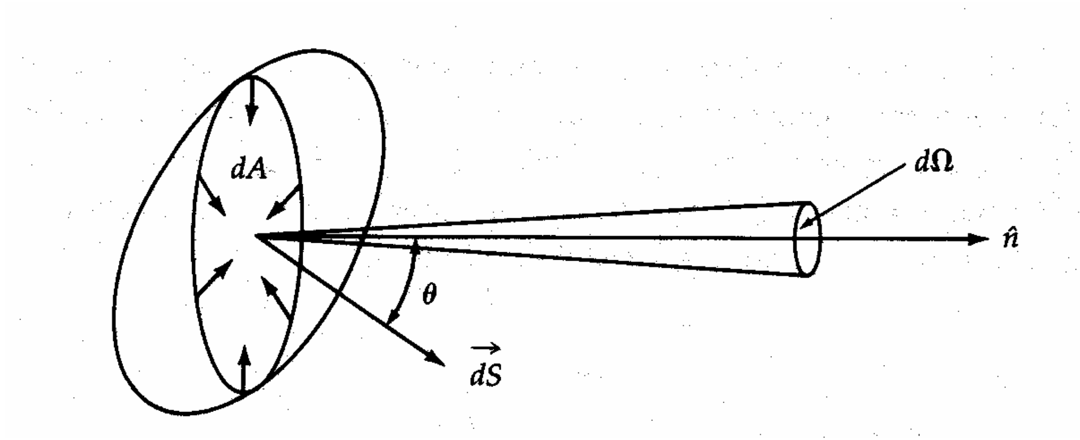

Figure 9.2 shows the differential parameters defining the specific intensity. Since dA is a differential area, the end of the differential solid angle dΩ covers it and all photons passing through dA in the direction $\hat{n}$ flow into dΩ.

We let the energy carried by photons with momentum p, moving in a direction$\hat{n}$, passing through a differential area dA, into a differential solid angle dΩ, in a time dt and frequency interval dν be d$E_\nu(p,\ \hat{n})$. We can then define the *specific intensity* as

$$I_\nu(p,\hat{n}) \equiv \frac{dE_\nu}{dA\cos\theta\, d\Omega\, d\nu\, dt} \tag{9.2.3}$$

Now the number of photons traveling in a direction $\hat{n}$ and crossing dA in a time dt comes from a physical volume

$$dV = c\,dA\cos\theta\, dt \tag{9.2.4}$$

However, the number of photons occupying that volume is just

$$dN = f(p,x)\,dV_p\,dV \tag{9.2.5}$$

For photons in that volume, there is no preferred direction so that the differential volume of momentum space is

$$dV_p = 4\pi p^2\, dp \tag{9.2.6}$$

[see equation (1.3.6)]. Some of these photons will flow in a direction $\hat{n}$, and into the differential solid angle dΩ, each carrying energy hν. Therefore, the differential energy in our definition of specific intensity becomes

$$dE_\nu = h\nu\, dN \frac{d\Omega}{4\pi} \tag{9.2.7}$$

Combining equations (9.2.2) through (9.2.7), we can relate the specific intensity to the phase space density of photons

$$I_\nu(p, \hat{n}) = \frac{h^4\nu^3}{c^2} f(p, x) \qquad f(p, x) = \frac{c^2}{h^4\nu^3} I_\nu(p, \hat{n}) \tag{9.2.8}$$

b General Equation of Radiative Transfer

Now let us rewrite equation (9.2.1) in vector form:

$$\frac{\partial f(p, \vec{r})}{\partial t} + \dot{\vec{r}} \cdot \nabla f + \dot{\vec{p}} \cdot \nabla_p f = S \tag{9.2.9}$$

If we assume that the photons are moving under the influence of a strong potential gradient $\nabla\Phi$, then we can write for photons that

$$\dot{\vec{r}} = c\hat{n} \qquad \vec{p} = \frac{h\nu}{c}\hat{n} \qquad \dot{\vec{p}} = \frac{h\nu}{c^2}\nabla\Phi \qquad \nabla_p = \frac{\hat{n}c\partial}{h\partial\nu} \tag{9.2.10}$$

Substitution of equations (9.2.10) and (9.2.8) into equation (9.2.9) yields an extremely general form of the equation of radiative transfer:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \nabla I_\nu + \hat{n} \cdot \nabla\Phi \left(\frac{\nu}{c^2}\right)\left(\frac{\partial I_\nu}{\partial \nu} - \frac{3I_\nu}{\nu}\right) = \frac{h^4\nu^3 S}{c^3} \tag{9.2.11}$$

This equation gives the correct description of the transfer of radiation in an arbitrary coordinate system, even if the boundary conditions are changing on a time scale comparable to the photon diffusion time. It is even correct if the photons are subject to energy loss by virtue of their moving through a strong gravitational field, although some care must be exercised in the choice of coordinates. However, if the propagation takes place in a dispersive medium, then $\dot{\mathrm{r}}$ must be replaced by

$$\dot{\vec{r}} = \frac{c}{\mathrm{n}}\hat{n} \tag{9.2.12}$$

and the unit of length is changed by n, where n is the index of refraction of the medium, as well.

Fortunately, in normal stellar atmospheres, the radiation field is time-independent, and the gravitational potential gradient is usually negligible so that equation (9.2.11) becomes

$$\hat{n} \cdot \nabla I_\nu = \frac{h^4 \nu^3 S}{c^3} \tag{9.2.13}$$

The assumption of plane parallelism will simplify this even further, but first let us turn to the "creation" rate *S*.

c "Creation" Rate and the Source Function

The "creation" rate *S* is just a measure of the rate at which photons that contribute to the flow through d*A* into d**Ω** are lost to the volume d*V*dV_p. Any absorption process that takes place in that phase space volume will result in the loss of a photon. However, photons can be "lost" from the volume without being destroyed. Any scattering process that changes the momentum of the photon can remove the photon from the volume. Thus, we can write the number lost to the differential volume as

$$dn_l = \alpha f \, dV \, dV_p \tag{9.2.14}$$

where α is just the fraction of particles present that are lost due to scattering and absorption. Particles may also "appear" in the volume or be "created" by thermal emission or scattering processes. We assume that the thermal emission processes are isotropic so that the number gained in the volume and radiated into a unit solid angle is

$$dn_{gt} = \frac{\epsilon \, dV \, dV_p}{4\pi} \tag{9.2.15}$$

where ε is the thermal emission per unit volume of phase space.

The situation for scattering is somewhat more complicated. Photons may appear in the volume and be scattered by matter in the volume into direction $\hat{n}$ with the appropriate momentum. These photons appear to be created just as surely as the thermal photons do, but with a difference. The thermal emission rate depends only on the thermodynamic characteristics of the material gas, whereas the scattered photons have their origin directly in the radiation field. This dependence of the "creation" rate, and hence the specific intensity, on the radiation field itself is one of the hallmarks of radiative transfer in stellar atmospheres. It is through the scattering process that the local value of the radiation field depends on the values of the radiation field throughout the medium. This coupling of the local radiation field to the global radiation field generates mathematical problems of an entirely different character from those found in stellar interiors.

Definition of the Redistribution Function For scattering to act as a source of photons in the direction and solid angle of interest, the process must take a photon of a given momentum and change its direction and momentum to coincide with that of the beam (i.e., the direction and frequency of the specific intensity). The processes that can do this are characterized by a function known as the redistribution function. This function is essentially the probability that a photon with a initial momentum p' coming from an initial solid angle $\mathbf{\Omega}'$ will be scattered into a solid angle $\mathbf{\Omega}$ with final momentum p. We call this probability density function $\mathrm{R}(p', p, \mathbf{\Omega}', \mathbf{\Omega})$ the *redistribution function* because it describes how interacting photons will be redistributed in momentum and direction. It is normalized so that

$$\int_0^\infty \int_0^\infty \oint_{4\pi} \oint_{4\pi} \mathrm{R}(p', p, \mathbf{\Omega}', \mathbf{\Omega})\, d\mathbf{\Omega}'\, d\mathbf{\Omega}\, dp'\, dp = 1 \tag{9.2.16}$$

The specific nature of the redistribution function depends on the details of the physical scattering mechanism and is discussed later. At this point, it is necessary only to know that the redistribution function exists and can be calculated for specific physical processes. Since the redistribution function has been normalized in accordance with equation (9.2.16), it represents the redistribution of a *scattered* photon. To calculate the number of particles gained from scattering, we still must include a measure of the fraction entering the volume $dVdV_p$ that undergo a scattering. Therefore, the number of particles gained from scattering processes is

$$dn_{gs} = \frac{\sigma'}{4\pi} \int_0^\infty \oint_{4\pi} \mathrm{R}(p', p, \mathbf{\Omega}', \mathbf{\Omega}) f(p', \mathbf{\Omega}')\, d\mathbf{\Omega}'\, dp'\, dV\, dV_p \tag{9.2.17}$$

where σ' is simply that fraction. The integrals run over all values of p' and $\mathbf{\Omega}'$ so that photons entering the volume from all possible directions and with all possible values of momentum are included.

"Creation" Rate in Terms of Scattering and Absorption Processes The net change of particles in volume $dVdV_p$ is obtained by combining equations (9.2.14), (9.2.15), and (9.2.17) and replacing the momentum derivatives and photon phase densities by equations (9.2.2) and (9.2.8). This process yields

$$\begin{aligned} dn &= dn_{gt} + dn_{gs} + dn_l \\ &= \frac{c^2}{h^4\nu^3}\left[\frac{h^4\nu^3}{c^2}\frac{\varepsilon}{4\pi} + \frac{\sigma' h}{4\pi c}\int_0^\infty \oint_{4\pi} \left(\frac{\nu}{\nu'}\right)^3 \mathrm{R}(\nu, \nu', \Omega', \Omega)\mathrm{I}_{\nu'}(\Omega')\, d\Omega'\, d\nu' - \alpha \mathrm{I}_\nu(\Omega)\right] dVdV_\mathrm{p} \end{aligned} \tag{9.2.18}$$

Now the "creation" rate S is the number of photons created per *unit* phase space volume per unit time. But in deriving the transformations from phase density to specific intensity given by equations (9.2.8), we did not choose an arbitrary spatial volume dV because it had a length cdt. Therefore, to relate the number of particles created in an arbitrary phase-space volume dVdV_p to S, we must normalize by that length so that

$$dn = \frac{S\,dV\,dV_p}{c}$$
(9.2.19)

Using this and equation (9.2.18) to express the "creation" rate S in terms of the physical processes taking place in the volume, we have

$$\frac{h^4\nu^3}{c^3} S = \rho j_\nu + \frac{\rho\sigma_\nu}{4\pi}\int_0^\infty \oint_{4\pi} R(\nu, \nu', \Omega, \Omega') I_{\nu'}(\Omega')\, d\Omega'\, d\nu' - (\kappa_\nu + \sigma_\nu)\rho I_\nu(\Omega)$$
(9.2.20)

where we have introduced the volume emissivity j_ν, the mass scattering coefficient σ_ν, and the mass absorption coefficient κ_ν and replaced R from

$$j_\nu = \frac{h^4\nu^3\epsilon/(4\pi c^2)}{\rho} \qquad \sigma_\nu = \frac{\sigma'}{\rho}$$

$$R(\nu, \nu', \Omega, \Omega') = \frac{h(\nu/\nu')^3}{c}\mathrm{R}(\nu, \nu', \Omega, \Omega')$$

$$\kappa_\nu + \sigma_\nu = \frac{\alpha}{\rho}$$
(9.2.21)

Thermal Emission For a gas that is in thermal equilibrium, the relationship between the rate of absorption and emission is not arbitrary. This is the first use of LTE. Since we are assuming that the gas is in thermal equilibrium with its surroundings (LTE), we may invoke Kirchhoff's law for the relationship between the thermal emissivity and absorptivity, namely,

$$j_\nu = \kappa_\nu B_\nu(T)$$
(9.2.22)

where B_ν(T) is the Planck function which depends only on the local temperature. If we use this and equation (9.2.20), the equation of radiative transfer given by equation (9.2.13) becomes

$$\hat{n}\cdot\nabla I_\nu = \rho\left[\kappa_\nu B_\nu(T) - (\kappa_\nu + \sigma_\nu)I_\nu(\Omega) + \frac{\sigma_\nu}{4\pi}\int_0^\infty \oint_{4\pi} R(\nu, \nu', \Omega, \Omega') I_{\nu'}(\Omega')\, d\Omega'\, d\nu'\right]$$
(9.2.23)

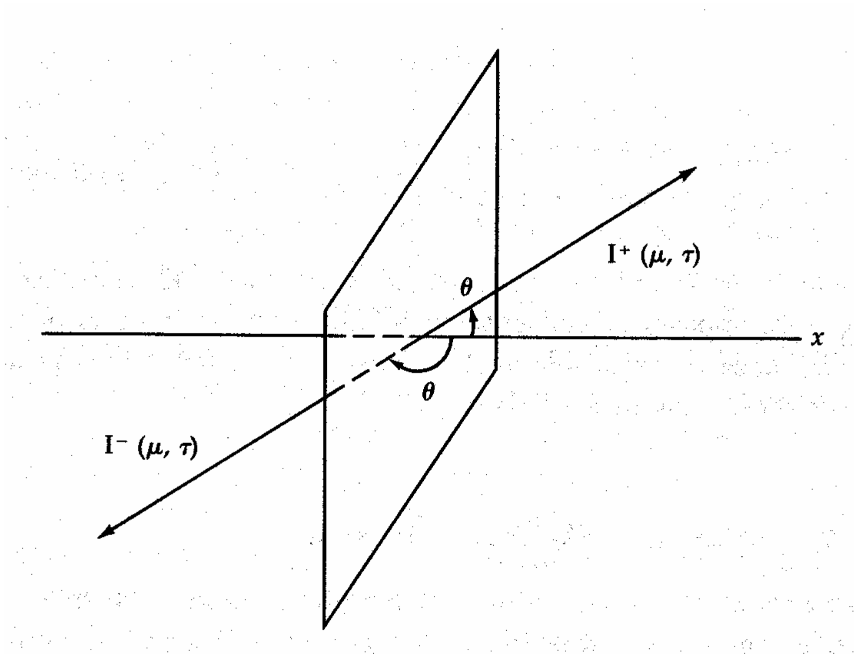

Figure 9.3 shows the geometry of a plane-parallel atmosphere.

We may further simplify the equation of radiative transfer by invoking the plane-parallel approximation so that ∇ becomes $\hat{x}d/dx$ (see Figure 9.3), yielding

$$\frac{\cos\theta}{-(\kappa_\nu + \sigma_\nu)\rho}\frac{dI_\nu}{dx} = I_\nu - \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} - \frac{\sigma_\nu}{4\pi(\kappa_\nu + \sigma_\nu)}\int_0^\infty \oint_{4\pi} R(\nu, \nu', \Omega, \Omega')I_{\nu'}(\Omega')\,d\Omega'\,d\nu' \tag{9.2.24}$$

Optical depth The notion of a dimensionless depth parameter called *optical depth* is central to the study of stellar atmospheres. It is usually taken to increase inward as one moves into the star, and it can be viewed physically in the following manner. Optical depth of unity is that depth of material wherein (1/*e*) of the photons will be scattered or absorbed while traversing the depth. In terms of the mass absorption and scattering coefficients and the differential distance parameter, it is defined as

$$d\tau_\nu = -(\kappa_\nu + \sigma_\nu)\rho\,dx \tag{9.2.25}$$

Making use of the definition of optical depth, we can write the equation of radiative transfer for a plane-parallel atmosphere as

$$\mu \frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - S_\nu(\mu, \tau_\nu) \tag{9.2.26}$$

where

$$S_\nu \equiv \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} + \frac{\sigma_\nu}{4\pi(\kappa_\nu + \sigma_\nu)} \int_0^\infty \oint_{4\pi} R(\nu, \nu', \Omega, \Omega') I_{\nu'}(\Omega')\, d\Omega'\, d\nu' \tag{9.2.27}$$

The parameter S_ν is known as the *source function* of the radiation field. Since the quantity $(\kappa_\nu+\sigma_\nu)$ appears so frequently, it is customary to call it the *mass extinction coefficient*. The name is reasonable as it is, indeed, a measure of the total ability of material to attenuate the flow of photons.

d Physical Meaning of the Source Function

The source function is one of the most important concepts in the theory of radiative transfer, and it is important to have a good intuitive feeling for its meaning. As the name implies, the source function represents the local contribution to the radiation field. It is a measure of the energy contributed to the radiation field by physical processes taking place at a particular spot in the atmosphere. Consider the case where scattering is unimportant so that $\sigma_\nu = 0$. Under these conditions the expression for the source function [equation (9.2.27)] becomes

$$S_\nu = B_\nu \tag{9.2.28}$$

and all photons locally contributed to the radiation field can be characterized by the Planck function since they arise from thermal processes. This is a consequence of the assumption of LTE which enabled us to use Kirchhoff's law to characterize the local emissivity of the gas in terms of its absorptivity. Some authors take this as a definition of LTE, but as such, it would be unduly restrictive. The presence of scattering, say by electrons will require a more complicated source function such as that given by equation (9.2.27), but the excitation and ionization characteristics of the gas may still be those expected for a gas in thermodynamic equilibrium. Thus, $S_\nu = B_\nu$ is normally a sufficient condition for the existence of LTE, but not a necessary one.

Now consider the case when pure absorption processes are negligible and virtually all the opacity of the material arises from scattering processes. Then

$$S_\nu = \frac{1}{4\pi} \int_0^\infty \oint_{4\pi} R(\nu, \nu', \Omega, \Omega') I_{\nu'}(\Omega')\, d\Omega'\, d\nu' \tag{9.2.29}$$

Here the source function depends only on the incident radiation field. Since the redistribution function is normalized to unity, the integral in equation (9.2.29) simply represents some sort of average of the local specific intensity over all frequencies and angles. The factor of $1/4\pi$ then represents that part of the average that is scattered into the differential solid angle appropriate for I_ν.

Thus, under the conditions of pure scattering, the source function becomes totally independent of the local physical conditions and is completely determined by the local radiation field. If this condition were to prevail throughout the atmosphere, one would have the curious result that the radiation field would be independent of the local values of the state variables (P, T, and ρ) and depend only on the ability of particles to scatter photons and the details of how the particles do it. In some real sense, the radiation field would become decoupled from the physical properties of the gas. Indeed, one can learn little about the physical conditions that prevail in a fog by observing the light transmitted through it from say an automobile headlight. This independence of the radiation field from the state variables of the gas enables one to solve the entire problem of radiative transfer for pure scattering without knowing anything about the gas other than the redistribution function. We use this property later to discuss methods of solving the equation of radiative transfer. However, as the case of the fog illustrates, this is a two edged sword. The decoupling of the radiation field from the state variables of the gas, in the case of pure scattering, means that we can not use the radiation field to determine the run of state variables with depth.

e Special Forms of the Redistribution Function

Since the redistribution function plays such an important role in specifying the nature of scattering in the source function, we examine some common physical situations and the corresponding redistribution functions.

Coherent Scattering The term *coherent scattering* refers to the case where photons are scattered in direction but not in frequency. Thomson scattering by electrons is of this form. Such processes are generally known as *conservative processes* because no energy is exchanged between the radiation field and the particles. While this is never strictly true, in many cases it is an excellent approximation. This is certainly true for the scattering of optical photons by the electrons present in a stellar atmosphere. Under these conditions we can write the redistribution function as

$$R(\nu', \nu, \vec{\Omega}, \vec{\Omega}') = h(\vec{\Omega}, \vec{\Omega}')\,\delta(\nu - \nu') \tag{9.2.30}$$

where $\delta(\nu-\nu')$ is the Dirac delta function. The delta function on frequency causes the frequency integral in equation (9.2.27) to collapse, simplifying the source function considerably.

Noncoherent Scattering This phrase has come to mean considerably more than the opposite of coherent scattering. For fully noncoherent scattering, the frequency of a scattered photon is completely uncorrelated with the frequency of the incident photon. In some sense, the photon "forgets" its prior frequency. Like coherent scattering, this case also represents an approximation. Clearly, if the situation were to apply to the entire frequency range from zero to infinity, the value of the redistribution function at any specific value of ν would have to be arbitrarily small. Thus, the common use of the approximation is confined to a finite frequency range such as a spectral line. As we shall see later, very strong spectral lines often possess the property that an electron in the upper state is so perturbed by interactions with other particles of the gas that the specific value of the absorbed energy is irrelevant in determining the energy of the photon that will be emitted in the subsequent transition. Thus, over a finite frequency interval, the wavelength of the emitted photon will be totally uncorrelated with the wavelength of the absorbed photon. Under these conditions, the frequency simply does not appear in the redistribution function and

$$R(\nu, \nu', \vec{\Omega}, \vec{\Omega}') = h(\vec{\Omega}, \vec{\Omega}') \tag{9.2.31}$$

Redistribution functions of this form are often called *complete redistribution functions*.

Isotropic Scattering As with complete redistribution, the photon undergoing isotropic scattering suffers from "amnesia". The direction of the scattered photon is completely uncorrelated with the direction of the incident photon. Thus, the angular dependence of the redistribution function vanishes and

$$R(\nu, \nu', \vec{\Omega}, \vec{\Omega}') = g(\nu, \nu') \tag{9.2.32}$$

This also considerably simplifies the source function in equation (9.2.27). If the radiation field were isotropic, the integral over the solid angle merely produces a factor of 4π, which cancels the corresponding factor in front of the integral. In general, this is also an approximation. Although it is far from obvious, we shall see that it is an excellent approximation for electron scattering of optical photons in a stellar atmosphere. So great is the simplification introduced by the assumption of isotropic scattering that there is a tendency to invoke it even when it is totally inappropriate. Later, we shall see what sorts of methods can be used to incorporate the full redistribution function in the solution of the equation of radiative transfer. Such cases are often called *partially coherent anisotropic scattering*, and their solution poses one of the most difficult problems in radiative transfer. However, before we consider these formidable problems, we must understand how to approach the solution of more basic problems. The dominant form of scattering in normal stellar atmospheres is Thomson scattering by electrons, and for purposes of determining the atmospheric structure it is an excellent approximation to assume that such scattering is isotropic. Under the assumption of coherent isotropic scattering, the source function given by equation

(9.2.27) becomes

$$S_\nu = \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} + \frac{\sigma_\nu}{4\pi(\kappa_\nu + \sigma_\nu)} \oint_{4\pi} I_\nu(\Omega')\, d\Omega' \tag{9.2.33}$$

9.3 Moments of the Radiation Field

In Chapter 1 we saw that a good deal of information was gleaned and simplification achieved by taking moments of the phase density of the particles that made up the gas in question. By such methods we were able to obtain equations for the continuity of matter and momentum and eventually to develop expressions for the hydrodynamic flow of a gas and hydrostatic equilibrium. The basic approach was to throw away information contained in the phase density by averaging it over some appropriate part of the phase space volume. That part of the volume was generally taken to be described by coordinates for which we did not require specific knowledge of the phase density. Since we were to invoke STE for the gas, we knew that the details of the velocity distribution could be ignored since in thermodynamic systems the velocity distributions are specified by a single parameter (the temperature) which is related to the mean velocity. Thus, averaging the phase density over velocity or momentum space made good sense.

We may expect the same sort of benefits by taking moments of the radiation field and particularly the specific intensity, for there is a simple relation [equation (9.2.8)] between the specific intensity and the phase density of photons. However, here we must be careful because it is the momentum distribution of photons in which we are interested so that averaging over momentum space would remove the very information we seek. We must look to other coordinates of phase space to find those which can be considered unimportant.

One of our initial assumptions is the atmosphere is well approximated by a plane-parallel slab. By symmetry, the radiation flow through such a slab will be isotropic about the normal to the slab. Hence, no important information will be contained in the azimuthal coordinate (see Figure 9.3). In addition, we might expect that information in the polar angle θ will not play a central role in the interaction of the radiation field with matter. It is this interaction that determines the emergent spectrum and the atmospheric structure. For these reasons, we can expect that the angular coordinates of phase space may prove expendable and that averages of the radiation field over these coordinates could prove useful in describing the flow of radiation through the atmosphere. Thus, we shall average over two of the three spatial coordinates, choosing the third to represent the direction of net energy flow. In the case of the plane-parallel atmosphere, this clearly is the direction of the atmosphere normal. Also, because of the simple transformation between the specific intensity and the photon phase density, the quantity to be averaged should be the specific intensity itself.

In addition, the higher-order moments should involve the spatial coordinates just as the higher moments in Chapter 1 involved the velocity itself. Such angular moments will then describe various aspects of the net flow of energy.

a Mean Intensity

Averaging over the angular coordinates described in Figure 9.3 is equivalent to averaging over all solid angles, so with some generality we can define the lowest-order moment of the radiation field as

$$J_\nu(\tau_\nu) \equiv \frac{\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\, d\Omega}{\oint_{4\pi} d\Omega} = \frac{1}{4\pi}\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\, d\Omega \tag{9.3.1}$$

For a plane-parallel atmosphere, where the intensity has no ϕ dependence and $\cos\theta$ is replaced by μ, equation (9.3.1) is equivalent to

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu, \tau_\nu)\, d\mu \tag{9.3.2}$$

This quantity, known as the *mean intensity*, is analogous to the particle density of Chapter 1 and differs from the photon energy density by a factor of $4\pi/c$.

b Flux

The next-highest-order moment is related to the net flow of energy in a specific direction $\hat{n}$, and it is defined, in a manner analogous to that for the mean intensity J_ν, as follows:

$$\vec{H}_\nu(\tau_\nu) = \frac{\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\hat{n}\, d\Omega}{\oint_{4\pi} d\Omega} = \frac{1}{4\pi}\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\hat{n}\, d\Omega \tag{9.3.3}$$

If we break $\hat{n}$ into its components, then for the axis-symmetric case of a plane parallel atmosphere, this becomes

$$\vec{H}_\nu(\tau_\nu) = \frac{\hat{n}}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\mu\, d\mu \tag{9.3.4}$$

where $\hat{n}$ points along the normal to the atmosphere. Indeed, it is fair to describe the flux as an intensity-weighted unit vector pointing in the direction of the flow of energy. Although the flux as defined here is a vector quantity, it is common to drop the vector properties since they are generally obvious from the geometry of the atmosphere. However, the vector nature does point to the similarity with the moments of defined in

Chapter 1 where the first moment of the phase density was the mean flow velocity. This definition of the first moment of the radiation field is sometimes known as the *Harvard flux* because it is heavily employed by the ATLAS atmosphere computer code developed at Harvard University, where the analogy to the mean intensity was deemed more important than the physical interpretation.

The actual energy crossing a differential area dA in the direction $\hat{n}$ is

$$\mathbf{F}_\nu(\tau_\nu) = \oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\, \hat{n}\, d\Omega = 4\pi H_\nu(\tau_\nu) \tag{9.3.5}$$

The quantity $\mathbf{F}_\nu$ is often called the *physical flux* because it represents the actual flow of energy. For a plane-parallel atmosphere this reduces to

$$\mathbf{F}_\nu(\tau_\nu) = 2\pi \int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\mu\, d\mu \tag{9.3.6}$$

The quantity π appears so regularly that many early authors, who were primarily concerned with plane-parallel atmospheres, defined a third form of the flux as

$$F_\nu(\tau_\nu) \equiv 2 \int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\mu\, d\mu = \frac{\mathbf{F}_\nu(\tau_\nu)}{\pi} = 4H_\nu(\tau_\nu) \tag{9.3.7}$$

This has become known as the *radiative flux* and it neither represents a physical quantity directly nor is analogous to the mean intensity. However, it is the most widely used definition of the first moment of the radiation field, so the student is to be warned to determine which definition of the flux a particular author is using or else all sorts of confusion may result. Throughout this book, we use all three definitions, but we try to be quite clear as to which is which and why a specific choice is made.

c Radiation Pressure

The analogy between this moment and the pressure tensor in Chapter 1 is very close, and the formal definition has the same normalization properties as J_ν. So

$$\mathbf{K}_\nu(\tau_\nu) = \frac{\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)(\hat{n}\hat{n})\, d\Omega}{\oint_{4\pi} d\Omega} = \frac{1}{4\pi}\oint_{4\pi} (\hat{n}\hat{n}) I_\nu(\tau_\nu, \theta, \phi)\, d\Omega \tag{9.3.8}$$

In a manner similar to the physical flux $\mathbf{F}_\nu$, $\mathbf{K}_\nu$ can be regarded as an intensity-weighted unit dyadic (not to be confused with the unit tensor **1** that has components δ_{ij}). Now $\mathbf{K}_\nu$ is known as the *radiation pressure tensor* and is completely analogous to the pressure tensor **P** that we obtained in Chapter 1 [equation (1.2.25)]. The meaning of the unit dyadic (in this case the vector outer product of a unit vector with itself) can be seen by writing out the various Cartesian components of $\mathbf{K}_\nu$ in spherical coordinates:

$$K_\nu = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\begin{bmatrix} \hat{i}\hat{i}\sin^2\theta\cos^2\phi & \hat{i}\hat{j}\sin^2\theta\cos\phi\sin\phi & \hat{i}\hat{k}\sin\theta\cos\theta\cos\phi \\ \hat{j}\hat{i}\sin^2\theta\sin\phi\cos\phi & \hat{j}\hat{j}\sin^2\theta\sin\ \phi & \hat{j}\hat{k}\sin\theta\cos\theta\sin\phi \\ \hat{k}\hat{i}\sin\theta\cos\theta\cos\phi & \hat{k}\hat{j}\sin\theta\cos\theta\sin\phi & \hat{k}\hat{k}\cos^2\theta \end{bmatrix} \tag{9.39}$$

$$\times I_\nu(\tau_\nu)\sin\theta\, d\theta\, d\phi$$

For the axis-symmetric case this becomes

$$J_\nu(\tau_\nu) \equiv \frac{\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\, d\Omega}{\oint_{4\pi} d\Omega} = \frac{1}{4\pi}\oint_{4\pi} I_\nu(\tau_\nu, \theta, \phi)\, d\Omega \tag{9.3.10}$$

or

$$\mathbf{K}_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{+1}\left[\frac{\hat{i}\hat{i}(1-\mu^2)}{2}, \frac{\hat{j}\hat{j}(1-\mu^2)}{2}, \hat{k}\hat{k}\mu^2\right] I_\nu(\tau_\nu, \mu)\, d\mu \tag{9.3.11}$$

Now consider the case where the radiation field is nearly isotropic so that we may expand $I_\nu(\tau_\nu, \mu)$ in a rapidly converging series as

$$I_\nu(\tau_\nu, \mu) = \sum_{i=0}^{\infty} I_i(\tau_\nu)\mu^i \tag{9.3.12}$$

where the lead term $I_0(\tau_\nu)$ is the dominant term. The components of the radiation pressure tensor then become

$$\mathbf{K}_\nu(\tau_\nu) = \begin{bmatrix} \dfrac{\hat{i}\hat{i}\sum_{i=0}^{\infty} I_{2i}(\tau_\nu)}{(2i+1)(2i+3)} \\ \dfrac{\hat{j}\hat{j}\sum_{i=0}^{\infty} I_{2i}(\tau_\nu)}{(2i+1)(2i+3)} \\ \dfrac{\hat{k}\hat{k}\sum_{i=0}^{\infty} I_{2i}(\tau_\nu)}{2i+3} \end{bmatrix} \approx \frac{1}{3} I_0(\tau_\nu)\mathbf{1} \tag{9.3.13}$$

Define the scalar moment $K_\nu(\tau_\nu)$ so that

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\mu^2\, d\mu \approx \frac{I_0(\tau_\nu)}{3} \tag{9.3.14}$$

The identity of this moment to the magnitude of the radiation pressure tensor in the case of near isotropy ensures that

$$\nabla \cdot \mathbf{K}_\nu(\tau_\nu) = \nabla K_\nu(\tau_\nu) \tag{9.3.15}$$

The isotropy condition was required in Chapter 1 in order for the divergence of the pressure tensor to be replaced by the gradient of the scalar pressure. Thus, in every sense of the word $K_\nu(\tau_\nu)$ may be considered to be related to the pressure of radiation. There remains only the problem of units. Since **P** represents the transfer of momentum across a surface, the exact relationship is

$$\mathbf{K}_\nu(\tau_\nu) = \frac{c\mathbf{P}_\nu(\tau_\nu)}{4\pi} \qquad P_r(\tau_\nu) = \frac{4\pi K_\nu(\tau_\nu)}{c} \tag{9.3.16}$$

Although these expressions give the correct formulation of the radiation pressure in terms of moments of the radiation field, it is important to remember that the radiation pressure is not identical to the force per unit area exerted by photons. That will involve the opacity, for to exert a force the photon must interact with the matter. In the stellar interior, this was no problem because the mean free path was so short as to guarantee that all photons would interact in a short distance. However, in a stellar atmosphere, this is no longer true for some of the photons escape. We return to this point when we consider the forces acting on the gas.

9.4 Moments of the Equation of Radiative Transfer

In Chapter 1 we saw that much useful information could be obtained about the gas by taking moments of the Boltzmann transport equation. The process always generated moments of phase density that were of one order higher than that used to generate the equation itself. Thus, to be useful, a relation between the higher-order moment and one of lower order had to be found. If this could be done, a self-consistent set of moment equations could be found and solved, yielding the values of those moments throughout the configuration. A similar set of circumstances will exist for the equation of radiative transfer.

To maintain a high level of generality, let us consider the general equation of radiative transfer given by equation (9.2.11) but with the "creation rate" replaced by the source function and the potential gradient taken to be zero. Thus

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \nabla I_\nu = -\rho(\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu) \tag{9.4.1}$$

Furthermore, assume that the scattering is isotropic and coherent so that the source function in equation (9.2.27) becomes

$$S_\nu = \frac{\kappa_\nu B_\nu}{\kappa_\nu + \sigma_\nu} + \frac{\sigma_\nu J_\nu}{\kappa_\nu + \sigma_\nu} \tag{9.4.2}$$

Now we integrate equation (9.4.1) over all solid angles, using the form of the source function given by equation (9.4.2), and get

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t} + \frac{1}{4}\nabla \cdot \vec{F}_\nu = \kappa_\nu \rho (B_\nu - J_\nu) \tag{9.4.3}$$

This is the equation of radiative equilibrium and describes how the radiative flux flows through the atmosphere. Note that the effects of scattering have disappeared from this equation. This is an expression of the conservative nature of scattering. Since no energy is gained or lost in each individual scattering event, the average can contribute nothing to the energy balance for the radiative flux and so all scattering terms must vanish.

a Radiative Equilibrium and Zeroth Moment of the Equation of Radiative Transfer

Consider the right-hand side of equation (9.4.3). This is essentially the right-hand side of the Boltzmann transport equation, which denotes the creation and destruction of particles in phase space, suitably averaged over direction. Thus, if there is no net production of photons in the atmosphere, this term, integrated over frequency, must be zero. Therefore, integrating equation (9.4.3) over all frequencies, we get

$$\frac{1}{c}\frac{\partial}{\partial t}\int_0^\infty J_\nu \, d\nu + \frac{1}{4}\nabla \cdot \int_0^\infty \vec{F}_\nu \, d\nu = 0 = \int_0^\infty \kappa_\nu \rho (B_\nu - J_\nu) \, d\nu \tag{9.4.4}$$

This is a very general statement of radiative equilibrium, and either side of this equation is an equivalent statement of it. If we let $F = \int_0^\infty F_\nu d\nu$, then for a static plane-parallel atmosphere

$$\frac{dF}{dx} = 0 \qquad \text{or} \qquad F = \text{const} = \frac{\sigma T_e^4}{\pi} \tag{9.4.5}$$

This will serve as a definition of the local effective temperature T_e.

b First Moment of the Equation of Radiative Transfer and the Diffusion Approximation

We multiply equation (9.4.1) [with the source function of equation (9.4.2) replacing the "creation rate"] by a unit vector $\hat{n}$ pointing in the direction of flow of radiant energy and integrate over all solid angles to obtain

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + 4\nabla \cdot \mathbf{K}_\nu = -\rho(\kappa_\nu + \sigma_\nu)\vec{F}_\nu \tag{9.4.6}$$

Now we make the approximation of near isotropy for the radiation field that was done in equation (9.3.12) and evaluate $J_\nu(\tau_\nu)$ from its definition [equation (9.3.1)] to get

$$J_\nu(\tau_\nu) = \sum_{i=0}^{\infty} \frac{I_{2i}(\mu, \tau_\nu)}{2i+1} \approx I_0 \tag{9.4.7}$$

We have already shown that under similar assumptions $K_\nu(\tau_\nu) = I_0/3$, so for conditions of near isotropy

$$K_\nu(\tau_\nu) \approx \frac{J_\nu(\tau_\nu)}{3} \tag{9.4.8}$$

This is known as the *diffusion approximation* and it can be used to close the moment equation (9.4.6), yielding

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + \frac{4}{3}\nabla J_\nu = -\rho(\kappa_\nu + \sigma_\nu)\vec{F}_\nu \tag{9.4.9}$$

Now equations (9.4.3) and (9.4.9) can be combined, by utilizing radiative equilibrium [equation (9.4.4)], to produce a "wave equation" for the radiative flux F

$$\frac{1}{3}\nabla(\nabla \cdot \vec{F}) - \frac{1}{c}\frac{\partial}{\partial t}\left[\rho \int_0^\infty (\kappa_\nu + \sigma_\nu)\vec{F}_\nu \, d\nu\right] - \frac{1}{c^2}\frac{\partial^2 \vec{F}}{\partial t^2} = 0 \tag{9.4.10}$$

which has all the properties of the usual wave equation. Such an equation is useful in solving problems in radiative transfer when the boundary conditions change on a time scale comparable to the photon diffusion time through the medium. Such situations may occur in some nebulae, novae and supernovae, or possibly quasars. For stellar atmospheres, the time-independent solutions will generally be sufficient. For a plane-parallel atmosphere in which the radiation field can be viewed as static, equations (9.4.3) and (9.4.9) become, respectively,

$$\frac{dF_\nu}{dx} = 4\kappa_\nu\rho(B_\nu - J_\nu) \qquad \frac{dJ_\nu}{dx} = -\frac{3}{4}(\kappa_\nu + \sigma_\nu)\rho F_\nu \tag{9.4.11}$$

That the static equations will be appropriate for normal stellar atmospheres becomes apparent when we consider that the diffusion time for a photon through a stellar atmosphere is only a few orders of magnitude times the light travel time. An atmosphere is a place from which photons escape after perhaps a few dozen interactions. Normal stars do not change on so short a time scale.

c Eddington Approximation

Although the diffusion approximation provides a method for closing the moment equations of the equation of radiative transfer, it does not allow the complete solution of the problem. The moment equations are, after all, differential equations and are subject to boundary conditions. Specification of these boundary conditions will provide a complete and unique solution for the radiation field. Sir Arthur Stanley Eddington suggested an additional approximation, inspired by the diffusion approximation that allows for the sufficient specification of boundary conditions to permit the solution of equations (9.4.11).

We consider the situation at the surface, and we assume the emergent radiation field to be isotropic. Since there is generally no incident radiation at the surface of a star, and using the condition of near isotropy given by equation (9.3.12) we get

$$J_\nu(0) = \frac{1}{2}\int_0^1 I(\mu, 0)\,d\mu = \frac{1}{2}\sum_{i=0}^{\infty}\frac{I_i(0)}{i+1} \approx \frac{1}{2} I_0(0)$$
$$F_\nu(0) = 2\int_0^1 I(\mu, 0)\mu\,d\mu = 2\sum_{i=0}^{\infty}\frac{I_i(0)}{i+2} \approx I_0(0) \tag{9.4.12}$$

Hence,

$$J_\nu(0) = \tfrac{1}{2}F_\nu(0) \tag{9.4.13}$$

This and the condition of radiative equilibrium given by equation (9.4.4) provide the two additional constraints necessary to solve equations (9.4.11). For the case of the gray atmosphere (see Section 10.2) a particularly simple solution is given by equation (10.2.15). Although the emergent radiation field is only approximately isotropic, it is the genius of this approximation that the errors introduced by the surface approximation are somewhat offset by the errors incurred by the assumption of the diffusion approximation. Thus, as we shall see later, the Eddington approximation produces solutions for the radiation field that are usually accurate to about 10 percent. As a result, the Eddington approximation is frequently used to solve problems in radiative transfer. To do better, we shall have to do a great deal more.

We have seen that it is possible to describe the flow of radiation through a stellar atmosphere. The derivation involves the same formalisms that we developed in Chapter 1 to describe the flow of matter. The resulting description of this flow is known as the equation of radiative transfer and it differs significantly from the simple result developed for the study of stellar interiors. The differences point up one of the central differences between stellar interiors and stellar atmospheres. Deep inside a star, the structure of the gas and radiation field is fully determined by the local values of the state variables of the gas. This is not the case in the stellar atmosphere. At any given point in the atmosphere, the local radiation field is composed of photons which originated in an environment that differed significantly from the local environment. Thus, the solution for the equation of radiative transfer locally will depend on the solution everywhere. This global nature of radiative transfer in a stellar atmosphere is one of the central differences between the interior and the outer layers of a star. We now turn our attention to solving the equation of radiative transfer.

Problems

1. Show that the specific intensity along a ray in empty space is constant.

2. Compute the specific intensity and the radiative flux at a distance r on the axis of an emitting disk having radius ρ and temperature T_e. Assume the disk to be located at r = 0.

3. Derive the equation of radiative transfer that is appropriate for spherical geometry. List carefully all the assumptions that you make.

4. Derive the plane-parallel equation of radiative transfer appropriate for a dispersive medium with an index of refraction n which is different from unity and which may vary with position.

5. Show that for any diagonal tensor **A**, in spherical coordinates,

$$(\nabla \cdot \mathbf{A})_r = \frac{\partial A_{rr}}{\partial r} + \frac{2A_{rr} - A_{\theta\theta} - A_{\phi\phi}}{r}$$

6. Use the above equation to show that if **K** is a diagonal tensor with all elements equal to K, then

$$(\nabla \cdot \mathbf{K})_r = \frac{\partial K}{\partial r} + \frac{3K - J}{r}$$

Here *K, J*, and **K** have their usual meanings for radiative transfer [see equations (9.3.1), (9.3.8), and (9.3.14)].

7. Derive equation (9.2.11) from equations (9.2.8) through (9.2.10). Show all your work.

8. Derive equation (9.2.18) from equations (9.2.2), (9.2.8), (9.2.14), (9.2.15), (9.2.17). Show all your work.

9. Derive equation (9.4.10) from first principles and axioms. Clearly list all assumptions that you make.

Supplemental Reading

A number of books provide an excellent description of the processes taking place in a stellar atmosphere. For excellent, clear, and correct definitions of the quantities that appear in the theory of radiative transfer see

Mihalas, D.: *Stellar Atmospheres*, 2d ed. W.H. Freeman, San Francisco, 1978, Chap. 1, pp. 1-18.

Strong insight into problems posed by scattering can be found in

Sobolev, V.V.: *A Treatise on Radiative Transfer*, (Trans.: S. I. Gaposchkin), D.Van Nostrand, Princeton N.J., 1963 Chap. 1, pp. 1 - 37.

An excellent overall statement of the problem can be found in

Mustel, E.R.: *Theoretical Astrophysics*, (Ed.: V.A. Ambartsumyan, trans. J.B. Sykes), Pergamon, New York, 1958, pp. 1-8.

To have some feeling for just how long people have been worrying about problems like these and to sample the physical insight of one of the most insightful men of the twentieth century, read

Eddington, A.S.: *On the Radiative Equilibrium of the Stars*, Mon. Not. R. astr. Soc., 77, 1916, pp. 16 - 35.