10

Solution of the Equation of Radiative Transfer

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One-half of the general problem of stellar atmospheres revolves around the solution of the equation of radiative transfer. Although equation (9.2.11) represents a very general formulation of radiative transfer, clearly the specific nature of the equation of transfer will depend on the geometry and physical environment of the medium through which the radiation flows. The nature of the physical medium will also influence the details of the source function so that the source function may depend on the radiation field itself. Thus, the mode of solution may be expected to be different for the different conditions that exist. However, the notion of plane parallelism is common to so many stars and other physical situations that we spend a significant amount of effort investigating the solution of the equation of transfer for plane-parallel atmospheres.

10.1 Classical Solution to the Equation of Radiative Transfer and Integral Equations for the Source Function

There are basically two schools of approach to the solution of the equation of transfer. One involves the solution of an integral equation for the source function, while the other deals directly with the differential equation of transfer. Both have their merits and drawbacks. Since both are widely used, we give examples of each. Both involve the classical solution, so that we begin the discussion with that solution.

a Classical Solution of the Equation of Transfer for the Plane-Parallel Atmosphere

The equation of transfer is a linear differential equation, which implies that a formal solution exists for the radiation field in terms of the source function. This linear property is a marked difference from the situation in stellar interiors where the structure equations were all highly nonlinear. Although under some conditions the solution [i.e., $I_\nu(\tau_\nu,\mu)$] itself is involved in the source function, this involvement is still linear. Let us consider a fairly general equation of radiative transfer for a plane-parallel atmosphere, but one where we may neglect time-dependent effects and the presence of the potential gradient on the radiation field.

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu, \mu) \tag{10.1.1}$$

Since this equation is linear in $I_\nu(\tau_\nu,\mu)$, we may write the complete solution as the sum of the solution to the homogeneous equation plus any particular solution. So let us choose as homogeneous and particular solutions

$$I_h(\mu, \tau_\nu) = c_1 e^{-\tau_\nu/\mu} + c_2 e^{+\tau_\nu/\mu} \qquad I_p(\mu, \tau_\nu) = f(\tau_\nu) e^{\tau_\nu/\mu} \tag{10.1.2}$$

Substitution into the equation of transfer places constraints on c_1 and $f'(\tau_\nu)$, namely

$$c_1 = 0 \qquad f'(\tau_\nu) = \frac{-S_\nu(\tau_\nu) e^{-\tau_\nu/\mu}}{\mu} \tag{10.1.3}$$

While we have assumed that the geometry of the atmosphere is plane-parallel, we have not yet specified the extent of the atmosphere. For the moment, let us assume that the atmosphere consists of a finite slab of thickness τ_0 (see Figure 10.1). The general classical solution for the plane-parallel slab is then

$$I_\nu(\mu, \tau_\nu) = c_2 e^{\tau_\nu/\mu} - \int S(t) e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu} \tag{10.1.4}$$

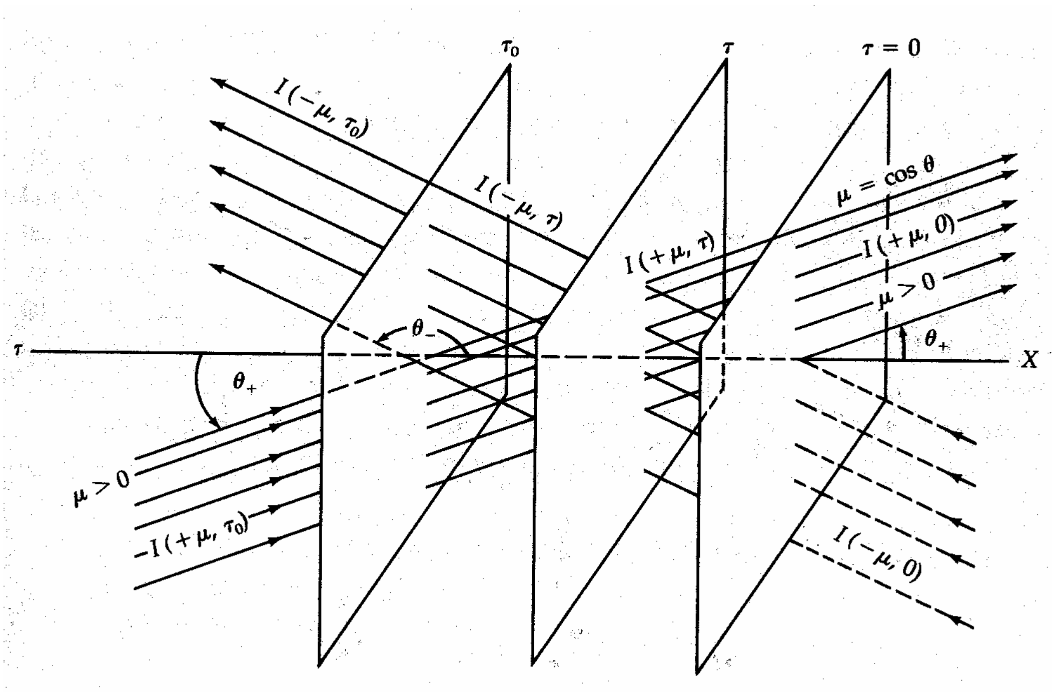

Figure 10.1 shows the geometry for a plane-parallel slab. Note that there are inward ($\mu<0$) and outward ($\mu>0$) directed streams of radiation. The boundary conditions necessary for the solution are specified at $\tau_\nu = 0$, and $\tau_\nu = \tau_0$.

Since the equation of transfer is a first order linear equation, only one constant must be specified by the boundary conditions. However, even though the depth variable τ_ν is the only independent variable that appears in a derivative, we must always remember that $I_\nu(\tau_\nu,\mu)$ is a function of the angular variable μ. Thus in general, the constant of integration c_2 will depend on the direction taken by the radiation. For radiation flowing outward in the atmosphere (that is, $\mu > 0$), the constant c_2 will be set equal to the radiation field at the base of the atmosphere [that is, $I_\nu(\mu, \tau_0)$] and the integral will include the contribution from the source function from all depths ranging from τ_0 to the point of interest τ_ν. If we were concerned about radiation flowing into the atmosphere (that is, $\mu < 0$), then the integral in equation (10.1.4) would cover the interval from 0 to τ_ν and c_2 would be chosen equal to the incident radiation field [$I_\nu(-\mu,0)$].

At this point we encounter one of the notational problems that often leads to confusion in understanding the literature in radiative transfer. For most problems in stellar atmospheres, there is a significant difference between the radiation field represented by the inward-directed streams of radiation and that represented by those flowing outward. In modeling the normal stellar atmosphere, there is no incident radiation present so that the incident intensity $I_\nu(-\mu,0) = 0$. However, the outward-directed streams always result from a lower boundary condition which is nonzero. Thus it is useful to distinguish between the inward- and outward-directed streams in some notational way. We have already used a standard method of indicating this difference; namely, we explicitly labeled the inward-directed streams by -μ. Thus, we usually regard the angular variable m as an intrinsically positive quantity that is bounded by $0 < \mu < 1$. The sign of m must then be explicitly indicated, and we do this when we use this convention. Thus, to gain a physical understanding of the meaning of any solution for the radiation field, one must always keep in mind which streams of radiation are being considered.

The general classical solution for the two streams can then be written as

$$\begin{aligned} I_\nu(+\mu, \tau_\nu) &= -\int_{\tau_0}^{\tau_\nu} S(t) e^{(\tau_\nu - t)/\mu} \frac{dt}{\mu} + I_\nu(+\mu, \tau_0) e^{(\tau_\nu - \tau_0)/\mu} \\ & \qquad\qquad \mu \ge 0 \\ I_\nu(-\mu, \tau_\nu) &= \int_0^{\tau_\nu} S(t) e^{-(\tau_\nu - t)/\mu} \frac{dt}{\mu} + I_\nu(-\mu, 0) e^{-\tau_\nu/\mu} \end{aligned} \tag{10.1.5}$$

While τ_ν represents the vertical depth in the atmosphere increasing inward, τ_ν/μ is the actual path along the direction taken by the radiation. In general, extinction by scattering or absorption will exponentially diminish the strength of the intensity by $e^{-\tau/\mu}$. Since the source function represents the local source of photons from all processes, and since it is attenuated by the optical distance along the path of the radiation, the integrand of the integral represents the local contribution of the source function to the value of the intensity at τ_ν. The remaining term simply represents the local contribution to the specific intensity of the attenuated incident radiation.

One further complication must be dealt with before we can use this description of a stellar atmosphere. In general, stellar atmospheres can be regarded as being infinitely thick. Since the influence of the lower boundary diminishes as $e^{\tau-\tau_0}$, and since this optical depth will exceed several hundred within a few thousand kilometers of the surface for main sequence stars, we can take it to be infinity. In addition, we should require the radiative flux to be finite everywhere. This will force the constant c_2 in equation (10.1.4) to vanish. Furthermore, the surface is generally unilluminated. So we can write the classical solution for the semi-infinite plane-parallel atmosphere as

$$\begin{aligned} I_\nu(+\mu, \tau_\nu) &= -\int_{\infty}^{\tau_\nu} S(t) e^{+(\tau_\nu - t)/\mu} \frac{dt}{\mu} \\ I_\nu(-\mu, \tau_\nu) &= \int_0^{\tau_\nu} S(t) e^{-(\tau_\nu - t)/\mu} \frac{dt}{\mu} \end{aligned} \tag{10.1.6}$$

b Schwarzschild-Milne Integral Equations

One reason that the equation of transfer admits such a simple solution compared to the equations of stellar structure is that we have confined most of the difficult physics to the source function. What is left is largely geometry and hence affords a simple solution. However, the classical solution does allow for the generation of the entire radiation field should it be possible to specify the source function. It also allows us to remove the explicit structure of the radiation field and to generate an expression for the source function itself. The result is an integral equation, that is, an equation where the unknown appears under the integral sign as well as outside it.

While much attention has been paid to the solution of differential equations, far less has been given to integral equations. However, it is very often numerically more efficient and accurate to solve an integral equation as opposed to the corresponding differential equation. Therefore, we spend some time and effort with these integral equations, for they provide a very productive path toward the solution of problems in radiative transfer.

Integral Equation for the Source Function In Chapter 9 we showed that, for coherent isotropic scattering, we could write a quite general expression for the source function [equation (9.2.33)]. If we re-express that result in terms of the mean intensity, we get

$$S_\nu(\tau_\nu) = \epsilon_\nu B_\nu[T(\tau_\nu)] + (1 - \epsilon_\nu)J_\nu(\tau_\nu) \tag{10.1.7}$$

where

$$\epsilon_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} \tag{10.1.8}$$

Now the role of the classical solution becomes evident. The source function contains the mean intensity $J_\nu(\tau_\nu)$, which can be generated from the classical solution that contains the source function itself. Thus, if we substitute the classical solution [equation (10.1.6)] into the definition for $J_\nu(\tau_\nu)$ [equation (9.3.2)], we get

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^1 I(+\mu', \tau_\nu)\,d\mu' + \frac{1}{2}\int_0^1 I(-\mu', \tau_\nu)\,d(-\mu')$$

$$J_\nu(\tau_\nu) = \frac{-1}{2}\int_0^1 \left[\int_\infty^{\tau_\nu} S(t)e^{+(\tau_\nu - t)/\mu'}\frac{dt}{\mu'}\right]d\mu'$$

$$+ \frac{1}{2}\int_0^1 \left[\int_0^{\tau_\nu} S(t)e^{-(\tau_\nu - t)/\mu'}\frac{dt}{\mu'}\right]d\mu' \tag{10.1.9}$$

Now notice that the argument of the exponential is always negative and that the two integrals over t are contiguous. Thus, we can combine these integrals into a single integral that ranges from 0 to 4. In addition, t and μ are independent variables so that we may interchange the order of integration and get

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S(t)\left[\int_0^1 e^{-|\tau_\nu - t|/\mu'}\frac{d\mu'}{\mu'}\right]dt \tag{10.1.10}$$

The quantity in brackets is a well-known function in mathematical physics known as the *exponential integral*. It depends only on the independent variables of the problem and therefore can be regarded as a largely geometric function. Its formal definition is

$$E_n(z) = \int_1^\infty \frac{e^{-zt}\,dt}{t^n} = \int_0^1 e^{-z/y}y^{n-2}\,dy = \int_0^1 \frac{e^{-z/y}y^{n-1}\,dy}{y} \tag{10.1.11}$$

and when expressed by the final integral, it has the same form as the integral in brackets in equation (10.1.10). While the exponential integral may not be terribly familiar, it should be regarded with no more fear and trepidation than sines and cosines. There is an entire set of these functions where each member is denoted by n, and they have a single argument, which for our purposes will be confined to the real line. These functions (except for the first exponential integral at the origin) are well behaved and resemble $e^{-x}/(nx)$ for large x. Some useful properties of exponential integrals are

$$\begin{aligned} nE_{n+1}(x) &= e^{-x} - xE_n(x) \qquad n > 1 \\ \frac{dE_{n+1}(x)}{dx} &= -E_n(x) \\ E_n(0) &= \frac{1}{n-1} \qquad E_n(\infty) = 0 \end{aligned} \tag{10.1.12}$$

Making use of the first exponential integral, we can rewrite our expression for the mean intensity [equation (10.1.10)] as

$$J_\nu(\tau_\nu) = \tfrac{1}{2}\int_0^\infty S(t)E_1|\tau_\nu - t|\,dt \tag{10.1.13}$$

Combining this with equation (10.1.7) for the source function, we arrive at the desired integral equation for the source function:

$$S_\nu(\tau_\nu) = \epsilon_\nu B_\nu[T(\tau_\nu)] + (1-\epsilon_\nu)\tfrac{1}{2}\int_0^\infty S_\nu(t)E_1|\tau_\nu - t|\,dt \tag{10.1.14}$$

Any function that multiplies the unknown in the integrand of an integral equation is called the *kernel* of the integral equation. Thus, the first exponential integral is the kernel of the integral equation for the source function. The connection between the physical state of the gas and the source function is contained in the term that makes the equation inhomogeneous, namely, the one involving the Planck function $B_\nu[T(\tau_\nu)]$. A solution of this equation, when combined with the classical solution, will yield the full solution to the radiative transfer problem since $I_\nu(\mu, \tau_\nu)$ will be specified for all values of μ and τ_ν.

It is possible to understand equation (10.1.14) from a physical standpoint. Now $\varepsilon(\tau_\nu)$ is the fraction of locally generated photons that arise from thermal processes, so that the first term is simply the local contribution to the source function from thermal properties of the gas. The second term represents the contribution from scattering. We have already said that a fundamental aspect of stellar atmospheres is the dependence of the local radiation field on the global solution for the radiation field. Nowhere is this more clearly demonstrated than in this term. The scattering contribution to the source function is made up of contributions from the source function throughout the atmosphere. However, these contributions decline with increasing distance from the point of interest, and they decline roughly exponentially.

One may object that this integral equation is a very specialized equation since it relies on the source function's being expressible in terms of the mean intensity and therefore is valid only for isotropic scattering. However, consider the very general expression for the source function given by equation (9.2.27). As long as the angular dependence of the redistribution function is known, it will be possible to carry out the integrals over solid angle and express the source function as a combination of the moments of the radiation field. As long as this can be done, the appropriate moments can be generated from the classical solution for the equation of transfer which will, in turn, involve only the source function. Thus, the moments can be eliminated from the moment expression for the source function, yielding an integral equation. To be sure, this will be a more complicated integral equation, but it will still be solvable by the same techniques that we apply to equation (10.1.14). Thus, the existence of an integral equation for the source function is a quite general result and represents the separation of the depth dependence of the radiation field from the angular dependence, which can be obtained from the classical solution.

Integral Equations for Moments of the Radiation Field Useful as the integral equation for the source function is, it is often convenient to have similar expressions for the moments of the radiation field. We should not be surprised that such expressions exist since the angular moments are free, by definition, of the angular dependence characteristic of the classical solution. Indeed, we have already supplied the required expressions to obtain an integral equation for the mean intensity. We simply use equation (10.1.7) to eliminate $S_\nu(t)$ from equation (10.1.13), and we have

$$J_\nu(\tau_\nu) = \tfrac{1}{2}\int_0^\infty \epsilon_\nu(t)B_\nu[T(t)]E_1|\tau_\nu - t|\,dt + \tfrac{1}{2}\int_0^\infty [1-\epsilon_\nu(t)]J_\nu(t)E_1|\tau_\nu - t|\,dt \tag{10.1.15}$$

It is now clear how to develop similar expressions for the remaining moments, since equation (10.1.13) was obtained by taking moments of the classical solution to the equation of transfer. Let us define an operator which is commonly used to represent this process.

$$\Lambda_n(\tau_\nu)|G(t)| \equiv \int_0^\infty \left(\frac{(t-\tau_\nu)}{|t-\tau_\nu|}\right)^{n+1} E_n|\tau_\nu - t|G(t)\,dt \tag{10.1.16}$$

The Λ_n operator is an integral operator which operates on a function by employing an exponential integral kernel. The term in large parentheses simply denotes the sign of the kernel throughout the region. With this integral operator, we can express the first three moments of the radiation field in terms of the source function as follows:

$$\begin{aligned} J_\nu(\tau_\nu) &= \tfrac{1}{2}\Lambda_1(\tau_\nu)|S_\nu(t)| \\ H_\nu(\tau_\nu) &= \tfrac{1}{2}\Lambda_2(\tau_\nu)|S_\nu(t)| \\ K_\nu(\tau_\nu) &= \tfrac{1}{2}\Lambda_3(\tau_\nu)|S_\nu(t)| \end{aligned} \tag{10.1.17}$$

Such equations are known as *Schwarzschild-Milne type of equations* and are extremely useful for the construction of model stellar atmospheres. For example, consider the condition of radiative equilibrium where it is necessary to know the radiative flux throughout the atmosphere, but not the complete radiation field. This information can be obtained directly with the aid of the flux equation of equations (10.1.17) and the source function. Thus, determination of the source function provides a complete solution of the radiative transfer problem.

c Limb-darkening in a Stellar Atmosphere

There is one property of the classical solution of the equation of transfer that we should address before moving on. If we consider the classical solution for the emergent intensity, we see that it basically represents the Laplace transform of the source function, namely

$$I(\mu, 0) = \int_0^\infty S(t)e^{-t/\mu}\frac{dt}{\mu} = \frac{\mathscr{L}[S(t)]}{\mu} \tag{10.1.18}$$

where $\mathscr{L}\,[S(\mathrm{t})]$ is the Laplace transform of the source function. Thus determination of the angular distribution of the emergent intensity is equivalent to determining the behavior of the source function with depth. Since the source function is determined by the temperature, determination of the depth dependence of the source function is equivalent to determining the depth dependence of the temperature. This is of considerable significance for stars where this dependence can be measured directly for it provides a direct observational check on the models of those stellar atmospheres.

If we anticipate some later results and assume that the source function can be approximated by

$$S(t) = at + b \tag{10.1.19}$$

then

$$I(\mu, 0) = a\mu + b \tag{10.1.20}$$

Thus, the coefficient *a* that multiplies the angular parameter μ in the emergent intensity is a direct measure of the source function gradient, while the constant term *b* denotes the value of the source function at the boundary. The decrease in brightness as one approaches the limb of the apparent stellar disk implied by equation (10.1.20) is called limb-darkening. Since for spherical stars the variation across the apparent disk is the same as the local angular dependence of the emergent intensity, measurement of the limb-darkening coefficient a yields a measurement of the source function gradient. This is of particular interest for the sun where such measurements are possible. Unfortunately, the poorest theoretical representation of the model atmosphere occurs near the surface, and this corresponds to just that region of the stellar disk (i.e., near the limb where $\mu \to 0$) where confirmatory measurements are most difficult to make. Although we have made an approximation to the depth dependence of the source function in equation (10.1.19), the approximation is unnecessary and more rigorous studies of this depth dependence would deal directly with the Laplace transform itself as given by equation (10.1.18). We have now compiled methods by which we can theoretically relate the emergent intensity to the source function and provided a potential observational method to verify our result. However, before discussing methods for the solution for the integral equation for the source function [equation (10.1.14)] we consider the solutions to a somewhat simpler problem, in order to gain an appreciation for the behavior of these solutions.

Empirical Determination of $T(\tau_\nu)$ for the Sun In the sun and some eclipsing binary stars, it is possible to determine the variation of the specific intensity across the apparent disk. If we approximate that variation by

$$\frac{I_\nu(0, \mu)}{I_\nu(0, 1)} \approx \sum_{i=0}^{n} a_i \mu^i \tag{10.1.21}$$

we can use equation (10.1.18) to obtain a power series representation of the source function with optical depth. Let us further assume that the source function can be represented by the Planck function, which in turn can be expanded in a power series in the optical depth so that

$$\frac{S_\nu(\tau_\nu)}{I_\nu(0,1)} \approx \frac{B_\nu[T(\tau_\nu)]}{I_\nu(0,1)} = \sum_{i=0}^{n} \beta_i \tau_\nu^i = \sum_{i=0}^{n} \beta_i \left(\frac{\tau_\nu}{\mu}\right)^i \mu^i \tag{10.1.22}$$

Then the substitution of this power series representation into equation (10.1.18) yields

$$\sum_{i=0}^{n} \beta_i i!\, \mu^i = \sum_{i=0}^{n} a_i \mu^i \tag{10.1.23}$$

Since for the sun, the a_i's and $I_\nu(0,1)$ may be determined from observation, the b_i's may be regarded as known. Thus, the temperature variation with monochromatic optical depth may be recovered from

$$\frac{2h\nu^3/c^2}{e^{h\nu/[kT(\tau_\nu)]} - 1} = I_\nu(0,1) \sum_{i=0}^{n} \beta_i \tau_\nu^i \qquad 10.1.24)$$

In the sun, the assumption that $S_\nu(\tau_\nu) = B_\nu(\tau_\nu)$ is a particularly good one, so that for the sun the optical depth variation of the temperature can be determined with the same sort of accuracy that attends the determination of the limb-darkening.

Empirical Determination of $\kappa(\tau_1) / \kappa(\tau_2)$ for the Sun This type of analysis can be continued under the above assumptions to obtain the variation with optical depth of the ratio of two monochromatic absorption coefficients. Since by definition

$$d\tau_\nu = -\kappa_\nu \rho\, dx \tag{10.1.25}$$

the ratio of two monochromatic optical depths is

$$\frac{d\tau_1}{d\tau_2} = \frac{\kappa_1(\tau_\nu)}{\kappa_2(\tau_\nu)} \tag{10.1.26}$$

Differentiating equation (10.1.22) with respect to temperature and substituting the result into equation (10.1.26), we get

$$\frac{\kappa_1(\tau_\nu)}{\kappa_2(\tau_\nu)} = \frac{I_2(0,1)}{I_1(0,1)} \frac{dB(\nu_1)/dT}{dB(\nu_2)/dT} \frac{\sum_i i\beta_i(\tau_2)\tau_2^{(i-1)}}{\sum_i i\beta_i(\tau_1)\tau_1^{(i-1)}} \tag{10.1.27}$$

Thus it is possible to determine the approximate wavelength dependence of the opacity for stars like the sun from the observed limb-darkening. Such observations provide a valuable check on the theory of stellar atmospheres.

10.2 Gray Atmosphere

For the better part of this century, theoretical astrophysicists have been concerned with the solution to an idealized radiative transfer problem known as the gray atmosphere. Although it is an idealized situation, it has some counterparts in nature. In addition, this problem possesses the virtue that a complete solution can be obtained for the radiation field without recourse to the physical details of the atmosphere. In this regard, the gray atmosphere model is rather like polytropic models for stellar interiors. As was the case for polytropes and stellar interiors, we may expect to gain significant insight into the properties of stellar atmospheres by understanding the solution to the gray atmosphere problem. The additional assumption required to turn our study of radiative transfer into that of a gray atmosphere is simple. Assume that the opacity, whether it is absorption or scattering, is independent of frequency. Thus, any frequency can be treated as any other frequency, as far as the radiative transfer is concerned. This independence of the radiative transfer from frequency has the interesting consequence that the mathematical solution to the equation of transfer for any frequency will be the solution for all frequencies, and thus must be the solution for the sum of all frequencies. Hence, the aspect of the solution that specifies the radiative flux also refers to the total flux, making the condition of radiative equilibrium relatively simple to apply. Since all aspects of the mathematical description are independent of frequency, we drop the subscript n for the balance of this discussion.

Knowing what we do about the physical processes of absorption, it is reasonable to ask if the gray atmosphere is anything more than an interesting mathematical exercise. Certainly bound-bound transitions are anything but gray. However, there are some bound-free transitions that exhibit only weak frequency dependence over substantial regions of the spectrum. If those regions of the spectrum correspond to that part of the spectrum containing most of the radiant flux, then the atmosphere will be very similar to a gray atmosphere. Absorption due to the H-minus ion is relatively frequency-independent throughout the visible part of the spectrum and in some stars is the dominant source of opacity. However, the premier example of a gray opacity source is electron scattering. Thomson scattering by free electrons is frequency-independent by definition, and for stars hotter than about 25,000 K, it is the dominant source of opacity throughout the range of wavelengths encompassing the maximum flow of energy. Thus, the early O and B stars have atmospheres that, to a very high degree, may be regarded as gray.

Since frequency dependence has been removed from the problem, we may write the equation of radiative transfer for a plane-parallel static atmosphere as

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - S(\tau) \tag{10.2.1}$$

where, for isotropic coherent scattering, the source function is

$$S(\tau) = \frac{\kappa B + \sigma J}{\kappa + \sigma} \tag{10.2.2}$$

Now the independence of the opacity on frequency makes the condition of radiative equilibrium given by equation (9.4.4) particularly simple.

$$\int_0^\infty \kappa_\nu \rho (B_\nu - J_\nu)\, d\nu = 0 = B - J \tag{10.2.3}$$

or simply

$$B = J \tag{10.2.4}$$

Substitution of this result into equation (10.2.2) yields

$$S(\tau) = B[T(\tau)] = J(\tau) \tag{10.2.5}$$

The fact that the mean intensity is equal to the Planck function and that either can be taken to be the source function has the interesting result that the solution to the gray atmosphere is independent of the relative roles of scattering and absorption. Thus, the radiation field for a pure absorbing gray atmosphere, where the source function is clearly the Planck function, will be indistinguishable from the radiation field of a pure scattering gray atmosphere. In addition, since there is a general independence on frequency, the spectral energy distribution will be that resulting from a gray atmosphere where the source function is the Planck function.

The gray atmosphere implies that all the development of Chapters 9 and 10 will apply at each frequency. This is indeed the easiest way to obtain equations (10.2.1) through (10.2.4). But there is much more. The integral equation for the source function [equation (10.1.14)] and that for the moments of the radiation field [equations (10.1.17)] become

$$\begin{aligned}
B(\tau) &= \tfrac{1}{2}\int_0^\infty B(t)E_1|t-\tau|\,dt \qquad J(\tau) = \tfrac{1}{2}\int_0^\infty J(t)E_1|t-\tau|\,dt \\
H(\tau) &= \tfrac{1}{2}\int_\tau^\infty B(t)E_2(t-\tau)\,dt - \tfrac{1}{2}\int_0^\tau B(t)E_2(\tau - t)\,dt \\
K(\tau) &= \tfrac{1}{2}\int_0^\infty B(t)E_3|t-\tau|\,dt
\end{aligned} \tag{10.2.6}$$

Solution of these equations, combined with the classical solution to the equation of transfer, yields a complete description for the radiation field at all depths in the atmosphere. The method of solution for the gray atmosphere equation of transfer is also illustrative of the methods of solution for the more general nongray problem.

a Solution of Schwarzschild-Milne Equations for the Gray Atmosphere

In general, an accurate solution of these equations must be accomplished numerically because the solution, even for the gray atmosphere, is not analytic everywhere. Particular care must be taken with these equations because the first exponential integral behaves badly as its argument approaches zero. Specifically

$$\lim_{x \to 0} E_1(x) = -\ln x \to \infty \tag{10.2.7}$$

Thus, the kernel of first two of equations (10.2.6) has a singularity when t = τ. However, this singularity is integrated over, and the integral is finite and well behaved. For years this singularity was regarded as an insurmountable barrier, and interest in the solution of the integral equations of radiative transfer languished in favor of more direct methods applicable to the differential equation of transfer itself. However, the singularity of the kernel is not an essential one and may be easily removed. Simply adding and subtracting the solution B(τ) from the right-hand side of the first of equations (10.2.6) yields

$$B(\tau) = \tfrac{1}{2}\int_0^\infty [B(t) - B(\tau)]E_1|t-\tau|\,dt + \tfrac{1}{2}B(\tau)\int_0^\infty E_1|t-\tau|\,dt \tag{10.2.8}$$

The integrand of the first of these integrals is now well-behaved for all values of (t) since [B(t)-B(τ)] will go to zero faster than the exponential integral diverges as $t \to \tau$. The only condition placed on the solution is that B(τ) satisfy a Lipschitz condition which is a weaker condition than requiring the solution to be continuous. The second integral is analytic and can be evaluated by using the properties of exponential integrals given in equations (10.1.12). This yields a slightly different integral equation, but one that has a well behaved integrand:

$$B(\tau) = \frac{\int_0^\infty [B(t) - B(\tau)]E_1|t-\tau|\,dt}{E_2(\tau)} \tag{10.2.9}$$

A simple way to deal with this type of integral equation is to replace the integral with some standard numerical quadrature formula. While Simpson's rule enjoys a great popularity, a gaussian-type quadrature scheme offers much greater accuracy for the same number of points of evaluation of the integrand. When the integral is so replaced, we obtain

$$B(\tau) = \frac{\sum_{i=1}^{n} [B(t_i) - B(\tau)]E_1|t_i - \tau|W_i}{E_2(\tau)} \tag{10.2.10}$$

which is a functional equation for B(τ) in terms of the solution at a discrete set of points t_i. The quantities W_i are just the weights of the quadrature scheme appropriate for the various points t_i. Evaluating the functional equation for B(τ) with τ equal to each value of t_j, and rearranging terms, we can obtain a system of linear algebraic equations for the solution at the specific points t_i:

$$\sum_{k=1}^{n} B(t_k)\left[\sum_{i=1}^{n} \frac{(\delta_{ik} - \delta_{jk})E_1|t_i - t_j|W_i}{E_2(t_j)} - \delta_{kj}\right] = 0 \qquad j = 1, \ldots, n \tag{10.2.11}$$

The term governed by the summation over i depends only on the type of quadrature scheme chosen, and so the equation (10.2.11) represents n linear homogeneous algebraic equations that have the standard form

$$\sum_{k=1}^{n} B(t_k)A_{kj} = 0 \qquad j = 1, \ldots, n \tag{10.2.12}$$

The fact that these equations are homogeneous points out an observation made earlier. For the gray atmosphere, the radiation field is decoupled from the values of the physical state variables. Thus, the homogeneous equations constitute an eigenvalue problem, and, as we see later, the eigenvalue is the value of the total radiative flux or alternately the effective temperature. One approach to the solution of equations (10.2.12) would be to define a new set of variables $B(t_i) / B(t_1)$ say, and to generate a system of inhomogeneous equations that can then be solved for the ratio of the source function to its value at one of the given points. Once the source function (or its ratio) has been found at the discrete points t_i, the solution can be obtained everywhere by substitution into equation 10.2.10. Since this is a functional equation, the results will have the same level of accuracy as that obtained for the values of $B(t_i)$. To achieve a level of accuracy significantly greater than that offered by the Eddington approximation, we will have to use a particularly accurate quadrature formula. Also the exponential nature of the exponential integral implies that the quadrature scheme should be chosen with great care.

b Solutions for the Gray Atmosphere Utilizing the Eddington Approximation

We have already seen that the diffusion approximation yields moment equations from the equation of transfer given by equation (9.4.11). For the gray atmosphere, these take the particularly simple form

$$\frac{dF}{d\tau} = 0 \qquad \frac{dJ}{d\tau} = \frac{3}{4}F$$

(10.2.13)

The first is a statement of radiative equilibrium which says that for a gray atmosphere F_ν is constant, and its integrated value can be related to the effective temperature. The second equation is immediately integrable, yielding a constant of integration. Thus,

$$F(\tau) = \text{const} = \frac{\sigma T_e^4}{\pi} \qquad J(\tau) = \tfrac{3}{4}F\tau + \text{const}$$

(10.2.14)

Using the Eddington approximation as given by equation (9.4.13), we can evaluate the constant and arrive at the dependence of the mean intensity with depth in the atmosphere.

$$J(0) = \tfrac{1}{2}F = \text{const} \qquad J(\tau) = \tfrac{3}{4}F(\tau + \tfrac{2}{3})$$

(10.2.15)

Remembering that J = *S* = B for a gray atmosphere in radiative equilibrium, we find that the temperature of the atmosphere should vary as

$$\left[\frac{T(\tau)}{T_e}\right]^4 = \tfrac{3}{4}(\tau + \tfrac{2}{3})$$

(10.2.16)

Thus, we see that at large depths, where we should expect the diffusion approximation to yield accurate results, the source function becomes linear with depth. Also, when τ = 2/3, the local temperature equals the effective temperature. So, in some real sense, we can consider the optical "surface" to be located at τ = 2/3. This is the depth from which the typical photon emerges from the atmosphere into the surrounding space. Only at depths less than 2/3 does the source function begin to depart significantly from linearity with depth. Unfortunately, this is the region in which most of the spectral lines that we see in stellar spectra are formed. Thus, we will have to pay special attention to that part of the atmosphere lying above optical depth 2/3.

We may check on the accuracy of the Eddington approximation by seeing how well it reproduces the surface boundary condition that it assumes. Using the definition for the mean intensity, the classical solution for the equation of transfer [equation (10.1.5)], and the fact that the source function is J itself, we obtain

$$J(0) = \frac{1}{2}\int_0^1 I(\mu, 0)\,d\mu = \frac{1}{2}\int_0^1\int_0^\infty J(t)e^{-t/\mu}\frac{dt}{\mu}\,d\mu = \frac{7F}{16}$$

(10.2.17)

So the Eddington approximation fails to be self-consistent by about 1 part in 8 or 12.5 percent in reproducing the surface value for the flux. To improve on this result, we will have to take a rather more complicated approach to the radiative problem.

c Solution by Discrete Ordinates: Wick-Chandrasekhar Method

The following method for the solution of radiative transfer problems has been extensively developed by Chandrasekhar[1] and we only briefly sketch it and its implications here. The method begins by noting that if one takes the source function to be the mean intensity J, then the equation of transfer can be written in terms of the specific intensity alone. However, the resulting equation is an integrodifferential equation. That is, the intensity, which is a function of the two variables μ and τ, appears differentiated with respect to one of them and is integrated over the other. Thus,

$$\mu \frac{dI(\mu, \tau)}{d\tau} = I(\mu, \tau) - \frac{1}{2} \int_{-1}^{+1} I(\mu', \tau)\, d\mu' \tag{10.2.18}$$

Now, as we did in the integral equation for the source function, we can replace the integral by a quadrature summation so that

$$\mu \frac{dI(\mu, \tau)}{d\tau} = I(\mu, \tau) - \frac{1}{2} \sum_{j=1}^{n} I(\tau, \mu_j) a_j \tag{10.2.19}$$

Here the a_j values are the weights of the quadrature scheme. This is a functional differential equation for $I(\tau,\mu)$ in terms of the solution at certain discrete values of μ_i. Chandrasekhar[1] is very explicit about using a gaussian quadrature scheme; a scheme that yields exact answers for polynomials of degree $2n - 1$ or less utilizes the zeros of the Legendre polynomials of degree *n* as defined in the interval -1 to +1. A more accurate procedure is to divide the integral in equation (10.2.18) into two integrals, one from -1 to 0 and the other from 0 to +1, and to approximate these integrals separately. The reason for this is that, since there is no incident radiation, the intensity develops a discontinuity in μ at τ = 0. Numerical quadrature schemes rely on the function to be integrated, in this case $I(\mu,\tau)$, being well approximated by a polynomial throughout the range of the integral. Splitting the integral at the discontinuity allows the resulting integrals to be well approximated where the single integral cannot be. This procedure is sometimes called *the double-gauss quadrature scheme*. However, this "engineering detail" in no way affects the validity of the basic approach.

As we did with equation (10.2.10), we evaluate the functional equation of transfer [equation (10.2.19)] at the same values of μ as are used in the summation so that

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - \frac{1}{2} \sum_{j=1}^{n} I(\tau, \mu_j) a_j \tag{10.2.20}$$

We now have a system of n homogeneous linear differential equations for the functions $I(\tau,\mu_i)$. Each of these functions represents the specific intensity along a particular direction specified by the value of μ_i. Since the value $\mu_i=0$ represents the point of discontinuity in $I(\mu,\tau)$ at the surface, this value should be avoided. Thus, there will normally be as many negative values of μ_i as positive ones. To solve the problem, we must find n constants of integration for the n first-order differential equations.

Inspired by the general exponential attenuation of a beam of photons passing through a medium, let us assume a solution of the form

$$I(\tau, \mu_i) = g_i e^{-k\tau} \tag{10.2.21}$$

Substitution of this form into this set of linear differential equations (10.2.20), will satisfy the equations if

$$\frac{S_\nu(\tau_\nu)}{I_\nu(0, 1)} \approx \frac{B_\nu[T(\tau_\nu)]}{I_\nu(0, 1)} = \sum_{i=0}^{n} \beta_i \tau_\nu^i = \sum_{i=0}^{n} \beta_i \left(\frac{\tau_\nu}{\mu}\right)^i \mu^i \tag{10.2.22}$$

and k satisfies the eigenvalue equation

$$1 = \sum_{j=1}^{n/2} \frac{a_j}{1 - \mu_j^2 k^2} \qquad \mu_j > 0 \tag{10.2.23}$$

Thus equation (10.2.22) provides a constant of integration for every distinct value of k. Since in all quadrature schemes the sum of the weights must equal the interval, $k^2 = 0$ will satisfy equation (10.2.23). Thus, since equation (10.2.23) is essentially polynomic in form there will be $n/2 - 1$ distinct nonzero values of k^2 and thus $n - 2$ distinct nonzero values of k which we denote as $\pm k_\alpha$. When these are combined with the value $k = 0$, we are still missing one constant of integration. Wick, inspired by the Eddington approximation, suggested a solution of the form

$$I(\tau, \mu_i) = b(\tau + q_i) \tag{10.2.24}$$

Substitution of this form into equations (10.2.20) also satisfies the equation of transfer provided that

$$q_i = \mu_i + Q \tag{10.2.25}$$

The product constant bQ can be identified with the constant obtained from k^2=0 so it cannot be regarded as a new constant of integration; but the term $b\tau$ can be regarded as such and therefore completes the solution, so that

$$I(\tau, \mu_i) = b\left[\sum_{\alpha=1}^{m}\left(\frac{L_{+\alpha}e^{-k_\alpha\tau}}{1+\mu_i k_\alpha} + \frac{L_{-\alpha}e^{+k_\alpha\tau}}{1-\mu_i k_\alpha}\right) + \mu_i + \tau + Q\right] \tag{10.2.26}$$

where

$$m = \frac{n}{2} - 1 \tag{10.2.27}$$

and the values of μ_i range from -1 to +1. The constants $L_{\pm\alpha}$ are the constants that result from equation (10.2.22) and the distinct values of k_α.

Moments of the Radiation Field from Discrete Ordinates We can generate the moments of the radiation field at a level of approximation which is consistent with the solution given by equation (10.2.26) by using the same quadrature scheme for the evaluation of the integrals over m that was used to replace the integral in the integrodifferential equation of radiative transfer. Thus,

$$J(\tau) = \tfrac{1}{2}\int_{-1}^{+1} I(\tau, \mu)\,d\mu = \tfrac{1}{2}\sum_{i=1}^{n} I(\tau, \mu_i)a_i \tag{10.2.28}$$

We already have the values I (τ,μ_i) required to evaluate the resulting sums. For the gaussian quadrature schemes suggested, the a_i's are symmetrically distributed in the interval -1 to +1, while the μ_i's are antisymmetrically distributed. Making use of these facts, substituting the solution [equation (10.2.26)] into equation (10.2.28), and manipulating, we get

$$J(\tau) = b\left[\sum_{\alpha=1}^{m}\left(L_{+\alpha}e^{-k_\alpha\tau} + L_{-\alpha}e^{+k_\alpha\tau}\right) + \tau + Q\right] \tag{10.2.29}$$

Following the same procedure for the flux, we get

$$F(\tau) = \frac{4b}{3} = \text{const} \tag{10.2.30}$$

so that the constant b of the Wick solution is related to the constant flux. All that remains to complete the solution is to determine the constants $L_{\forall\alpha}$ from the boundary conditions.

Application of Boundary Values to the Discrete Solution At no point in the derivation have we used of the fact that the atmosphere is assumed semi-infinite. So, in principle, the solution given by equation (10.2.26) is correct for finite slabs. Some applications of the approach have been used in the study of planetary atmospheres, and so for generality let us consider the application to an atmosphere which has a finite thickness τ_0. For such an atmosphere, we must know the distribution of the intensity entering the atmosphere at the base τ_0 as well as that which is incident on the surface. Given that, it is a simple matter to equate the solution [equation (10.2.26)] to the boundary values, and we get

$$
\begin{aligned}
I(-\mu_i, 0) &= \frac{3}{4} F \left[\sum_{\alpha=1}^{m} \left(\frac{L_{+\alpha}}{1-\mu_i k_\alpha} + \frac{L_{-\alpha}}{1+\mu_i k_\alpha} \right) - \mu_i + Q \right] \\
I(+\mu_i, \tau_0) &= \frac{3}{4} F \left[\sum_{\alpha=1}^{m} \left(\frac{L_{+\alpha} e^{-k_\alpha \tau_0}}{1+\mu_i k_\alpha} + \frac{L_{-\alpha} e^{+k_\alpha \tau_0}}{1-\mu_i k_\alpha} \right) + \mu_i + \tau_0 + Q \right] \\
i &= 1, \ldots, \frac{n}{2}
\end{aligned}
\qquad (10.2.31)
$$

These equations represent n equations in n unknowns. There are $2n$-2 values of $L_{\pm\alpha}$'s, F, and Q all specified by the n values of the boundary intensity. Here we explicitly incorporated the sign of μ_i into the equation so that all values of μ_i should be taken to be positive. Although the equations are effectively linear in the unknowns, note that the coefficients of those equations grow exponentially with optical depth. Indeed, since the nonzero values of k_α are all greater than unity, that growth is quite rapid. In practice, it is virtually impossible to solve these equations for any value of $\tau_0 > 100$. Indeed, if the order of approximation is large, the practical upper limit is nearer 10. This instability is inherent in all discrete ordinate methods used for finite atmospheres.

The reason is fairly straightforward. Each of the k_α's corresponds to a stream of radiation with a particular value of μ_i. The total optical path for this radiation stream is τ_0/μ_i. Since the solution of equation (10.2.26) is essentially a linear two-point boundary-value problem, the solution at one boundary is determined by the solution at the other boundary. If part of the solution at one boundary is optically remote from the other boundary, it will decouple from the solution, causing the solution to become singular or poorly determined. Physically, the photons from the remote boundary have been so randomized by scatterings or absorptions that all information pertaining to their direction of entrance into the atmosphere has been lost. In the case of the semi-infinite atmosphere, this has explicitly been taken into account, and the information from the lower boundary is contained in the finite and constant radiative flux.

We can see the effect of this constraint on the discrete solution by examining the behavior of the solution [equation (10.2.31)] as $\tau_0 \rightarrow 4$. Since we require the radiation field to remain finite as $\tau_0 \rightarrow 4$, the $L_{-\alpha}$'s must go to zero. Thus, the influence of the deep radiation field explicitly disappears from the solution, and the radiative flux becomes the eigenvalue of the problem. So the complete solution for the semi-infinite gray atmosphere for the method of discrete ordinates is

$$B(\tau) = J(\tau) = \frac{3}{4} F\left(\sum_{\alpha=1}^{m} L_{+\alpha} e^{-k_\alpha \tau} + \tau + Q\right)$$

$$I(-\mu_i, 0) = 0 = \frac{3}{4} F\left(\sum_{\alpha=1}^{m} \frac{L_{+\alpha}}{1 - \mu_i k_\alpha} - \mu_i + Q\right)$$

$$F = \text{const} = \frac{\sigma T_e^4}{\pi}$$

$$1 = \sum_{j=1}^{m+1} \frac{a_j}{1 - \mu_j^2 k_\alpha^2} \tag{10.2.32}$$

Table 10.1 contains some values of $L_{+\alpha}$, k_α, and Q for various orders of approximation for the semi-infinite gray atmosphere for the single-gauss quadrature scheme. By analogy to the Eddington approximation, the source function is sometimes written as

$$J(\tau) = \tfrac{3}{4} F[\tau + q(\tau)] \tag{10.2.33}$$

where

$$q(\tau) = Q + \sum_{\alpha=1}^{m} L_{+\alpha} e^{-k_\alpha \tau} \tag{10.2.34}$$

is known as the *Hopf function*. It is clear that for the Eddington approximation the appropriate Hopf function would be $q(\tau) = 2/3$. The Eddington approximation also avoids the problem of the solution's becoming unstable with increasing depth, by the use of the diffusion approximation, which basically assumes that the radiation field has been directionally randomized.

Nonconservative Gray Atmospheres The notion of a nonconservative gray atmosphere may sound like a contradiction in terms, and if it were meant to apply to all frequencies, it would be. However, consider the case where the opacity is essentially gray over the part of the spectrum containing most of the emergent radiation, but radiative equilibrium does not apply because some energy is lost from the radiation field to perhaps convection. Or consider an atmosphere where the dominant opacity source is the scattering of light from a hot external source, but the atmosphere itself is so cold that the thermal emission can be neglected. Planetary atmospheres often fit into this category.

Table 10.1 Values of the Eigenvalues k_α and Integration Constants L_α and Q for the Semi-infinite Plane-Parallel Gray Atmosphere

		k_α			
$n/2$	2	4	10	20	
Q	0.69402480	0.70691789	0.70991539	0.71031562	/α
	1.972027	1.103185	1.012222	1.002743	1
	xxxxxxxx	1.591779	1.054805	1.012774	2
19	−2.0932	4.458086	1.133904	1.028846	3
18	−1.7809		1.263385	1.052985	4
17	−1.5440		1.471808	1.085545	5
16	−1.3454		1.822337	1.127770	6
15	−1.1717		2.481124	1.181417	7
14	−1.0164		4.059775	1.248953	8
13	−0.87584	xxxxxxxx	12.068353	1.333858	9
12	−0.74792			1.441246	10
11	−0.63132			1.578567	11
10	−0.52526			1.757429	12
9	−0.42926	−3.9483	xxxxxxxx	1.996629	13
8	−0.34305	−2.9333		2.328706	14
7	−0.26646	−2.2033		2.815209	15
6	−0.19945	−1.6167		3.588119	16
5	−0.14198	−1.1331		4.990712	17
4	−0.094949	−0.73841		8.281846	18
3	−0.055675	−0.42775	−8.3920	24.791774	19
2	−0.026843	−0.20005	−3.6186	xxxxxxxx	
1	−0.0078584	−0.055516	−0.94609	−11.667	
α	20	10	4	2	$n/2$
		$L_{+\alpha} \times 10^2$			

Under these conditions, the equation of transfer becomes

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{p}{2} \int_{-1}^{+1} I(\tau, \mu')\, d\mu' \tag{10.2.35}$$

which, by the same methods used to generate equation (10.2.23), yields the eigenvalue equation

$$1 = p \sum_{j=1}^{n/2} \frac{a_j}{1 - \mu_j^2 k_\alpha^2} \qquad \mu_j > 0 \tag{10.2.36}$$

Here p is the scattering albedo, or the fraction of interacting photons that are scattered. Since $p < 1$ for a nonconservative atmosphere, there will now be n distinct k_α's and n distinct $L_{\pm\alpha}$'s, so that the n values of the boundary radiation field completely specify the solution. The $L_{\forall\alpha}$'s are specified by the boundary equations

$$I(-\mu_i, 0) = \sum_{\alpha=1}^{n/2} \left(\frac{L_{+\alpha}}{1 - \mu_i k_\alpha} + \frac{L_{-\alpha}}{1 + \mu_i k_\alpha} \right)$$
$$I(+\mu_i, 0) = \sum_{\alpha=1}^{n/2} \left(\frac{L_{+\alpha}}{1 + \mu_i k_\alpha} + \frac{L_{-\alpha}}{1 - \mu_i k_\alpha} \right) \qquad (10.2.37)$$

and the source function for the atmosphere is given by

$$S(\tau) = J(\tau) = \sum_{\alpha=1}^{n/2} L_{+\alpha} e^{-k_\alpha \tau} + L_{-\alpha} e^{+k_\alpha \tau} \qquad (10.2.38)$$

We need not consider the unilluminated semi-infinite atmosphere since all radiation moving up through a nonconservative semi-infinite atmosphere will eventually be lost before it emerges. Thus, only the finite slab or an illuminated semi-infinite nonconservative atmosphere will yield anything other than the trivial solution.

10.3 Nongray Radiative Transfer

While the elimination of the assumption of a gray opacity removes the easy incorporation of radiative equilibrium into the solution of the equation of radiative transfer, most methods described in Section 10.2 can be used for the nongray case. In spite of the diversity of methods available to the researcher for the solution of radiative transfer problems (there are more than are described here), most practical approaches can be divided into two categories: the solution of the integral equation for the source function and methods based on the solution of the differential equations for the radiation field. The solution of the integral equation for the source function is highly efficient, since no more information is generated than is necessary for the solution of the problem, and has also proved effective in dealing with problems of polarization, where complex redistribution functions are required (see Chapter 16). The differential equation approach is perhaps more widely used because a highly efficient algorithm has been developed which enables the investigator to utilize existing and proven mathematical packages for much of the numerical work. In addition, the differential equation approach has proved effective where geometries other than plane-parallel ones are required, and lends itself naturally to the incorporation of time-dependent and hydrodynamic terms when they may be needed. Of the myriads of specific applications, we will be concerned with only two.

a Solutions of the Nongray Integral Equation for the Source Function

We derived the integral equation for the nongray source function in Section 10.1 [equation (10.1.14)]. The approach we take is basically that described for the solution of the Schwarzschild-Milne equations in Section 10.2. Replacing the integral in equation (10.1.14) with a suitable quadrature scheme, after removing the singularity of the first exponential integral as described in equation (10.2.8), we get

$$S_\nu(\tau_\nu) = \epsilon_\nu B(\tau_\nu) + \tfrac{1}{2}(1-\epsilon_\nu)\left\{\sum_{j=1}^{n} [S_\nu(t_j) - S_\nu(\tau_\nu)]E_1|t_j - \tau_\nu|W_j + S_\nu(\tau_\nu)[2 - E_2(\tau_\nu)]\right\} \tag{10.3.1}$$

This functional equation, evaluated at the points of the quadrature, yields a set of linear algebraic equations for the source function at the quadrature points. These, in turn, can be put into standard form so that

$$\sum_{k=1}^{n} S_\nu(t_k)\left\{\tfrac{1}{2}\sum_{j=1}^{n} [1-\epsilon(t_i)][(\delta_{kj} - \delta_{ki})E_1|t_i - t_k|W_i + \delta_{ik}E_2(t_i)] + \epsilon(t_i)\delta_{ik}\right\} = \epsilon_\nu(t_i)B_\nu(t_i) \qquad i = 1, \ldots, n \tag{10.3.2}$$

These equations are strongly diagonal since the dominant contribution to the source function is always the local one. That contribution is measured by the last term in equation (10.3.1), and it represents the addition made to the equation to compensate for the removal of the local contribution within the integral. The strongly diagonal nature of the equations ensures that the solution is numerically stable. Indeed, when $S_\nu = B_\nu$ and $\varepsilon_\nu = 1$, the equations are formally diagonal. Thus, in practice they may be solved rapidly by means of the Gauss-Seidel iteration with $S_\nu(t_i) = B_\nu(t_i)$ as the initial guess. We remarked earlier that some care should be taken in choosing the quadrature scheme. It is a good practice to split the integral into two parts, with the first ranging from 0 to 1 and the second from 1 to 4. A 10-point Gauss-Legendre quadrature provides sufficient accuracy for the rapid change of the source function near the surface, while a 4-point Gauss-Laguerre quadrature scheme is adequate for the second as the source function approaches linearity.

A slightly different approach is taken by the Harvard group[2] in the widely used atmosphere program called ATLAS. They also solve the integral equation for the source function, but they deal with the singularity of the exponential integral in a somewhat different fashion. Instead of formally removing the singularity, they approximate the source function with cubic splines over a small interval. With an analytic form for the source function, it is possible to evaluate the integral, resulting in a multiplicative weight for the coefficients of the splines. This results in a series of weights which are numerically very similar to those present in equation (10.3.2). Again, a set of linear algebraic equations is produced for the source function at a discrete set of optical depths. The results of the two methods are nearly identical, with the gaussian quadrature scheme being somewhat more efficient.

b Differential Equation Approach: The Feautrier Method

This method replaces the differential equations of radiative transfer with a set of finite difference equations for parameters related to the specific intensity at a discrete set of values for the angular variable m_i. However, the choice of values of m_i is irrelevant to understanding the method, so we leave that choice arbitrary for the moment. Instead of solving the equation of transfer for the specific intensity, we write equations of transfer for combinations of inward- and outward-directed streams.

Feautrier Equations Consider the variables

$$u \equiv \tfrac{1}{2}[I(+\mu,\tau_\nu) + I(-\mu,\tau_\nu)] \qquad v \equiv \tfrac{1}{2}[I(+\mu,\tau_\nu) - I(-\mu,\tau_\nu)] \tag{10.3.3}$$

Here we have paired the outward directed stream I(+μ,τ) with its inward -directed counterpart I(-μ,τ) into quantities that resemble a "mean" intensity u and a "flux" v. One of the benefits of the linearity of the equation of transfer is that we can add or subtract such equations and still get a linear equation. Thus, by adding an equation for a +μ stream to one for a -μ stream we get

$$\mu\frac{dv}{d\tau_\nu} = u - S \qquad \mu > 0 \tag{10.3.4}$$

Similarly, by subtracting one from the other, we get

$$\mu\frac{du}{d\tau_\nu} = v \qquad \mu > 0 \tag{10.3.5}$$

Using this result to eliminate v from equation (10.3.4), we have

$$\mu^2\frac{d^2u}{d\tau_\nu^2} = u - S \qquad \mu > 0 \tag{10.3.6}$$

This is a second-order linear differential equation, so we will need two constraints or constants of integration. At the surface I(-μ,0) = 0, so v(0) = u(0), and from equation (10.3.5) we have

$$\mu\frac{du}{d\tau_\nu}\bigg|_{\tau_\nu=0} = u(0) \tag{10.3.7}$$

The other constraint on the differential equation comes from invoking the diffusion approximation at large depths. Under this assumption

$$I_\nu \simeq J_\nu \simeq B_\nu \simeq S_\nu \tag{10.3.8}$$

We may now use the equation of transfer itself to generate a perturbation expression for $I(\mu,\tau_\nu)$ at large depths:

$$\begin{aligned} I_\nu(\mu, \tau_\nu) &= \mu \frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} + S_\nu(\tau_\nu) \\ &\approx \mu \frac{dB_\nu(\tau_\nu)}{d\tau_\nu} + B_\nu(\tau_\nu) \qquad \tau_\nu \gg 1 \end{aligned} \tag{10.3.9}$$

Substituting the left-hand side into the definition for v we get

$$v = \mu \frac{dB_\nu(\tau_\nu)}{d\tau_\nu} = \mu \frac{du}{d\tau_\nu} \qquad \tau_\nu \gg 1 \tag{10.3.10}$$

Equations (10.3.7) and (10.3.10) are the two constraints needed to specify the solution. Now consider the finite difference approximations required to solve equation (10.3.6) subject to these constraints.

Solution of the Feautrier Equations We saw earlier, in Chapter 4, how the method of solution used to solve the Schwarzschild equations of stellar structure was supplanted by the Henyey method utilizing finite differences. Many of the reasons that lead to the superiority of the Henyey method are applicable to the Feautrier method for solution of the equations of radiative transfer. It is for that reason that we describe the numerical method in some detail.

First, we must pick a set of τ_k's for which we desire the solution. We must be certain that the largest τ_N is deep enough in the atmosphere to ensure that the assumptions resulting in the boundary condition given in equation (10.3.9) are met. In addition, it is useful if the density of points near the surface is large enough that the solution will be accurately described. This is particularly important when we are dealing with the transport of radiation within a spectral line. Now we define the following finite difference operators:

$$\begin{aligned} \Delta f_{k+1/2} &\equiv f(\tau_{k+1}) - f(\tau_k) \\ \Delta\tau_{k+1/2} &\equiv \tau_{k+1} - \tau_k \\ \Delta\tau_k &\equiv \tfrac{1}{2}(\Delta\tau_{k+1/2} + \Delta\tau_{k-1/2}) \end{aligned} \tag{10.3.11}$$

The subscript k+½ simply means that this is an estimate of the parameter appropriate for the value of τ midway between k and k + 1. Unlike the Henyey scheme, where this information was obtained from an earlier model structure, the Feautrier method obtains the information by linear interpolation from the existing solution. Now we replace the derivatives with the following finite difference operators:

$$\left.\frac{df(\tau)}{d\tau}\right|_{\tau=\tau_k} \simeq \frac{\Delta f_{k+1/2}}{\Delta\tau_{k+1/2}}$$
$$\left.\frac{d^2f(\tau)}{d\tau^2}\right|_{\tau=\tau_k} \simeq \frac{\Delta f_{k+1/2}/\Delta\tau_{k+1/2} - \Delta f_{k-1/2}/\Delta\tau_{k-1/2}}{\Delta\tau_k} \tag{10.3.12}$$

The second derivative in equation (10.3.6) can now be replaced by these operators operating on u(μ) to yield the following linear algebraic equations for u(μ) at the chosen optical depth points τ_k:

$$\mu^2 \frac{u_{k-1}(\mu)}{\Delta\tau_k \Delta\tau_{k-1/2}} - \frac{\mu^2}{\Delta\tau_k}\left(\frac{1}{\Delta\tau_{k-1/2}} + \frac{1}{\Delta\tau_{k+1/2}}\right)u_k(\mu) + \frac{\mu^2 u_{k+1}(\mu)}{\Delta\tau_k \Delta\tau_{k+1/2}}$$
$$= u_k(\mu) - S_k \tag{10.3.13}$$

Now it is time to pick those values of μ for which we desire the solution. Let u be considered a vector whose elements are the values of u at the particular values of μ_i, so that

$$\vec{u} = [u(\mu_1), u(\mu_2), u(\mu_3), \ldots, u(\mu_n)] \tag{10.3.14}$$

The linear equations (10.3.13) can now be written as a system of matrix-vector equations of the form

$$\mathbf{A}_k \cdot \vec{u}_{k-1} - \mathbf{B}_k \cdot \vec{u}_k + \mathbf{C}_k \cdot \vec{u}_{k+1} = -\vec{S}_k \tag{10.3.15}$$

The elements of matrices **A**, **B**, and **C** involve only the values of μ_i and τ_k that were chosen to describe the solution. The constraints given by equations (10.3.7) and (10.3.9) require that

$$\mathbf{A}_1 = \mathbf{0} \qquad \mathbf{C}_N = \mathbf{0} \tag{10.3.16}$$

Thus we have set the conditions required to solve the equations for $u_k(\mu_i)$ from which the specific intensity can be recovered and all the moments that depend on it. Equations (10.3.16) happen to be tridiagonal, which ensures that they can be solved efficiently and accurately. We have glossed over the source function S_k in our discussion by assuming that it is known everywhere and depends only on τ_k. However, the property of the source function that caused so many problems for earlier methods (and, indeed, resulted in the integral equations in the first place) is that the source function usually depends on the intensity itself. However, for scattering, the source function does depend on the intensity in a linear manner. Therefore, it is possible to represent the source function in terms of the unknowns u_k and v_k and include them in the equations, still preserving their tridiagonal form.

There is one caveat to this. The Feautrier method imposes a certain symmetry on the solution to the radiative transfer problem by combining inward- and outward-directed streams. If the redistribution function does not share this symmetry, it will not be possible to represent the scattering in terms of the functions $u(\mu)$ and $v(\mu)$. Thus, for some problems involving anisotropic scattering, the Feautrier method may not be applicable. In addition, when the redistribution function involves redistribution in frequency, the optical depth points must be chosen so that the deepest point will satisfy the assumptions required for the approximation given in equation (10.3.9) for all frequencies. If this is not done, errors incurred at those frequencies for which the assumptions fail can propagate in an insidious manner throughout the entire solution.

The Feautrier method does not suffer from the exponential instabilities described for the discrete ordinate method, because it invokes the diffusion approximation at large depths (specifically the inner boundary). The diffusion approximation basically contains the information that the radiation field has been randomized in direction and thereby stabilizes the solution in the same manner as it stabilizes the Eddington solution. As we see in Chapter 11, knowledge of the mean intensity, the radiative flux, and occasionally the radiation pressure is usually sufficient to calculate the structure of the atmosphere. The Feautrier method finds more information than that and therefore is not as efficient as it might be. However, the numerical methods for solving the resulting linear equations are so fast that the overall efficiency of the method is quite good, and it provides an excellent method of solution for most problems of radiative transfer in stellar atmospheres. Remember that, like any numerical method, the Feautrier method should be used with great care and only on those problems for which it is suited.

10.4 Radiative Transport in a Spherical Atmosphere

Any discussion of the solution of radiative transfer problems would be incomplete without some mention of the problem introduced by a departure from the simplifying assumption of plane-parallel geometry. In addition, there are stars for which the plane-parallel approximation is inappropriate, and we would like to model these stars as well as the main sequence stars for which the plane-parallel approximation is generally adequate. The density in the outer regions of red supergiants is so low that the atmosphere will occupy the outer 30 percent to 40 percent of what we would like to call the radius of the star. Here, the plane-parallel assumption is clearly inappropriate for describing the star. We must include the curvature of the star in any description of its atmosphere. In doing so, we will require a parameter that was removed by the plane-parallel assumption - the *stellar radius*. This parameter can be operationally defined as the distance from the center to some point where the radial optical depth to the surface is some specified number (say unity). In doing so, we must remember that the radius may now become a wavelength-dependent number and so some mean value from which the majority of the energy escapes to the surrounding space may be appropriate for describing the star when a single value for the radius is required. However, for the calculation of the stellar interior, we need to know only the surface structure at a given distance from the center in order to specify the interior structure. Whether the distance corresponds to our idea of a stellar radius is irrelevant. In addition, we assume that the star is spherically symmetric.

a Equation of Radiative Transport in Spherical Coordinates

In Chapter 9 we developed a very general equation of radiative transfer which was coordinate-independent [equation (9.2.11)]. Writing the time-independent form for which the gravity gradient does not significantly affect the photon energy, we get

$$\hat{n} \cdot \nabla I_\nu = \rho(\kappa_\nu + \sigma_\nu)(S_\nu - I_\nu)$$

(10.4.1)

Writing the ∇ operator in spherical coordinates and making the usual definition for m (see Figure 10.2), we get

$$\mu \frac{\partial I_\nu(r, \mu)}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I_\nu(r, \mu)}{\partial \mu} = \rho(\kappa_\nu + \sigma_\nu)[S_\nu(r, \mu) - I_\nu(r, \mu)]$$

(10.4.2)

where we take the source function to be that of a nongray atmosphere with coherent isotropic scattering, so that

$$S_\nu = \epsilon_\nu B_\nu[T(r)] + (1 - \epsilon_\nu)J_\nu(r)$$

(10.4.3)

Our approach to the solution of the equation of transfer will be to obtain and solve some equations for the important moments of the radiation field.

Radiative Equilibrium and Moments of the Radiation Field For a steady state atmosphere, our condition for radiative equilibrium [equation (9.4.4)] becomes

$$\nabla \cdot \int_0^\infty \vec{F}_\nu \, d\nu = 4 \int_0^\infty \kappa_\nu \rho(B_\nu - J_\nu) \, d\nu = 0$$

(10.4.4)

However, in spherical coordinates, the divergence of the total flux yields the same condition that we obtained for stellar interiors [equation (4.2.1)]:

$$\pi F \equiv \pi \int_0^\infty F_\nu \, d\nu = \frac{L}{4\pi r^2} \tag{10.4.5}$$

Now the condition of radiative equilibrium is obtained from the zeroth moment of the equation of transfer [equation (9.4.3)], while the first moment of the equation of transfer [equation (9.4.6)] yields an expression for the radiation pressure tensor. For an atmosphere with no time-dependent processes, these moment equations become

$$\nabla \cdot \vec{F}_\nu = 4\kappa_\nu \rho (B_\nu - J_\nu) \qquad \nabla \cdot \mathbf{K}_\nu = -\frac{\rho(\kappa_\nu + \sigma_\nu)\vec{F}_\nu}{4} \tag{10.4.6}$$

Noting that there is no net flow of radiation in either the θ or ϕ coordinates for a spherically symmetric atmosphere, we see that the divergence of the flux in spherical coordinates becomes

$$\frac{\partial(r^2 F_\nu)}{\partial r} = 4r^2 \kappa_\nu \rho (B_\nu - J_\nu) \tag{10.4.7}$$

If we make the assumption that the radiation field is nearly isotropic, then $\nabla \cdot \mathbf{K}_\nu$ becomes ∇K_ν where K_ν is the scalar moment that we have identified with the radiation pressure [see equations (9.3.14) through (9.3.16)]. Perhaps the easiest way to find the representation of equation (10.4.6) in spherical coordinates is to multiply equation (10.4.2) by m and integrate over all m. This yields the second of the required moment equations,

$$\frac{\partial K_\nu}{\partial r} + \frac{3K_\nu - J_\nu}{r} = -\frac{\rho(\kappa_\nu + \sigma_\nu)F_\nu}{4} \tag{10.4.8}$$

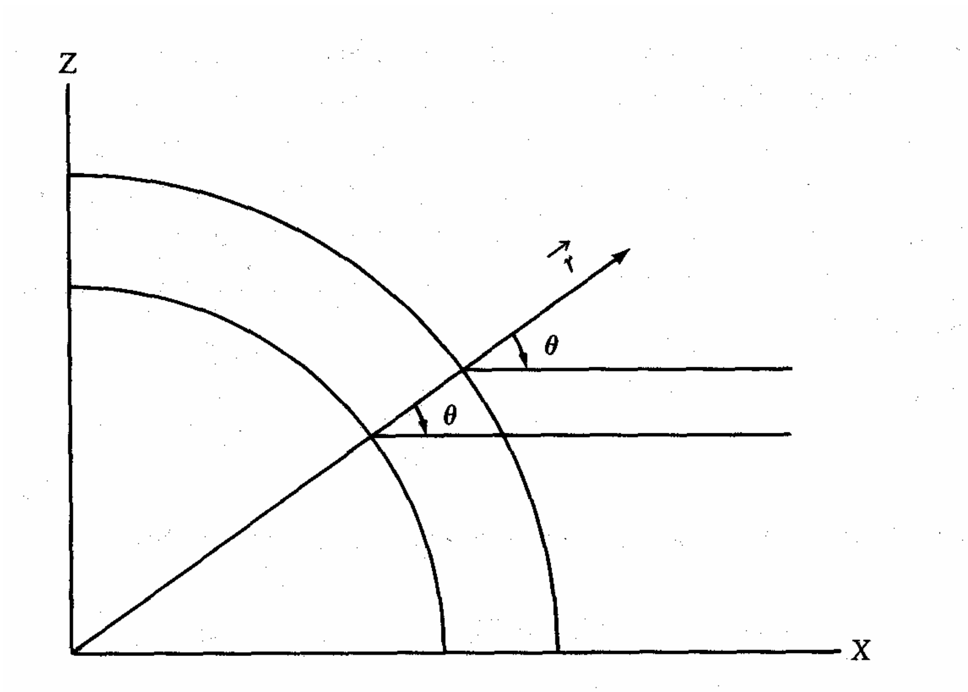

Figure 10.2 shows the geometry assumed for the Spherical Equations of radiative transfer. The angle θ for which $\mu = \cos\theta$ is defined with respect to the radius vector. Unlike the plane-parallel approximation the depth variable is the radius and increases outward.

Closing the Moment Equations and the Eddington Factor In Chapter 9 we observed [equation (9.4.8)] that under conditions of near isotropy $K_\nu = J_\nu/3$. This was the moment approximation needed to close the moment equations, and it is known as the *diffusion approximation*. However, such conditions do not prevail throughout the atmosphere, so it is common to assume that the two moments can be related by a scale factor, which has come to be known as the *Eddington factor,* defined as

$$f_\nu(r) = \frac{K_\nu(r)}{J_\nu(r)} \tag{10.4.9}$$

We can replace the radiation pressure by the Eddington factor and obtain

$$\frac{\partial(f_\nu J_\nu)}{\partial r} + \frac{(3f_\nu - 1)J_\nu}{r} = \frac{-\rho(\kappa_\nu + \sigma_\nu)F_\nu}{4} \tag{10.4.10}$$

for the second moment equation.

Equation (10.4.10) combined with equation (10.4.7) form a complete system for F_ν and J_ν subject to the appropriate boundary conditions. Of course, we have not fundamentally changed the problem since the Eddington factor is unknown and presumably a function of depth. It must be found so that any atmosphere produced is self-consistent under the constraint of radiative equilibrium. The Eddington factor basically measures the isotropy of the radiation field, since for isotropic radiation it is 1/3. Imagine a radiation field entirely directed along $\mu = \pm 1$. For such a field $f_\nu = 1$, while for a radiation field confined to a plane that is normal to this direction, $f_\nu = 0$. If we consider the normal radiation field emerging from a star, the temperature gradient normally produces limb-darkening, implying that the radiation field near the surface becomes more strongly directed along the normal to the atmosphere. Thus, we should expect the Eddington factor to increase as the surface approaches. This effect should be enhanced for stars with large spherical atmospheres. Thus, for normal stellar atmospheres

$$\tfrac{1}{3} \le f_\nu(r) < 1 \tag{10.4.11}$$

b An Approach to Solution of the Spherical Radiative Transfer Problem

Sphericality Factor This factor is introduced purely for mathematical convenience and as such has no major physical importance. However, it does tend to make the spherical moment equations resemble their plane-parallel counterparts. We define

$$\ln(r^2 q_\nu) \equiv \int_{r_c}^{r} \frac{3f_\nu - 1}{x f_\nu}\, dx + \ln r_c^2 \tag{10.4.12}$$

so that

$$\frac{-4}{\rho(\kappa_\nu + \sigma_\nu) q_\nu} \frac{\partial(r^2 q_\nu f_\nu J_\nu)}{\partial r} = r^2 F_\nu \tag{10.4.13}$$

The parameter r_c is the deepest radius for which the problem is to be solved. Given F_ν, we can find the sphericality factor q_ν by numerically integrating equation (10.4.12). Using this definition of q_ν, we may rewrite the second moment equation (10.4.10), as

$$\frac{-4}{\rho(\kappa_\nu + \sigma_\nu) q_\nu} \frac{\partial(r^2 q_\nu f_\nu J_\nu)}{\partial r} = r^2 F_\nu \tag{10.4.14}$$

This form is suitable for combining with the first moment equation (10.4.7), to eliminate F_ν and get

$$\frac{\partial^2(r^2 q_\nu f_\nu J_\nu)}{\partial \tau_\nu^2} = \frac{r^2 \epsilon_\nu (J_\nu - B_\nu)}{q_\nu} \tag{10.4.15}$$

where

$$\partial\tau_\nu = -q_\nu\rho(\kappa_\nu + \sigma_\nu)\,\partial r \tag{10.4.16}$$

and ε_ν has the same meaning as before [see equation (10.1.8)]. We have now generated a second order differential equation for J_ν that is similar to the one obtained for the Feautrier method, and we solve it in a similar manner.

Boundary Conditions The boundary conditions are determined in much the same manner as for the Feautrier method. For the lower boundary we make the same assumptions of isotropy as were made for equation (10.3.9). Indeed, we multiply equation (10.3.9) by μ and integrate over all μ, to get

$$F_\nu \simeq 2\int_{-1}^{+1} \mu^2 \frac{dB_\nu}{d\tau_\nu}\,d\mu + 2\int_{-1}^{+1} \mu B_\nu\,d\mu = \frac{4}{3}\frac{dB_\nu}{d\tau_\nu} \tag{10.4.17}$$

This and equation (10.4.14) allow us to specify the derivative of J_ν at the lower boundary as

$$\left.\frac{\partial(r^2 q_\nu f_\nu J_\nu)}{\partial\tau_\nu}\right|_{r=r_c} = \left.\frac{r_c^2}{3}\frac{dB_\nu}{d\tau_\nu}\right|_{r=r_c} \tag{10.4.18}$$

Again r_c is the deepest point for which the solution is desired. Equation (10.4.14) also sets the upper boundary condition at R as

$$4\left.\frac{\partial(r^2 q_\nu f_\nu J_\nu)}{\partial\tau_\nu}\right|_{r=R} = R^2 F_\nu = R^2 \frac{\int_0^1 \mu I(R,\mu)\,d\mu}{\int_0^1 I(R,\mu)\,d\mu} J_\nu(R) \tag{10.4.19}$$

so that we again have a two-point boundary-value problem and a second-order differential equation for J_ν which we can solve by the same finite difference techniques that were used for the Feautrier method [see equations (10.3.11) through (10.3.16)].

The problem can now be solved, assuming we know the behavior of the Eddington factor with depth in the atmosphere. Unfortunately, to find this, we must know the angular distribution of the radiation field at all depths. Normally, we could appeal to the classical solution, for knowledge of J_ν would provide all the information needed to calculate the source function. But the classical solution was appropriate for only the plane-parallel approximation. To find the analog for spherical coordinates, we have to use the symmetry of a spherical atmosphere and perform still another coordinate transformation.

Impact Space and Formal Solution for the Spherical Equation of Radiative Transfer Consider a coordinate frame attached to the star so that the z axis points in the direction of the observer and passes through the center of the star (see Figure 10.3). Coordinates z and *p* designate all places within the star with *p* playing the role of an impact parameter for photons directed toward the observer parallel to z. The entire solution set I(μ,r) can be represented by the radiation streams $I_+(p,z)$, and $I_-(p,z)$ by moving along surfaces of constant r.

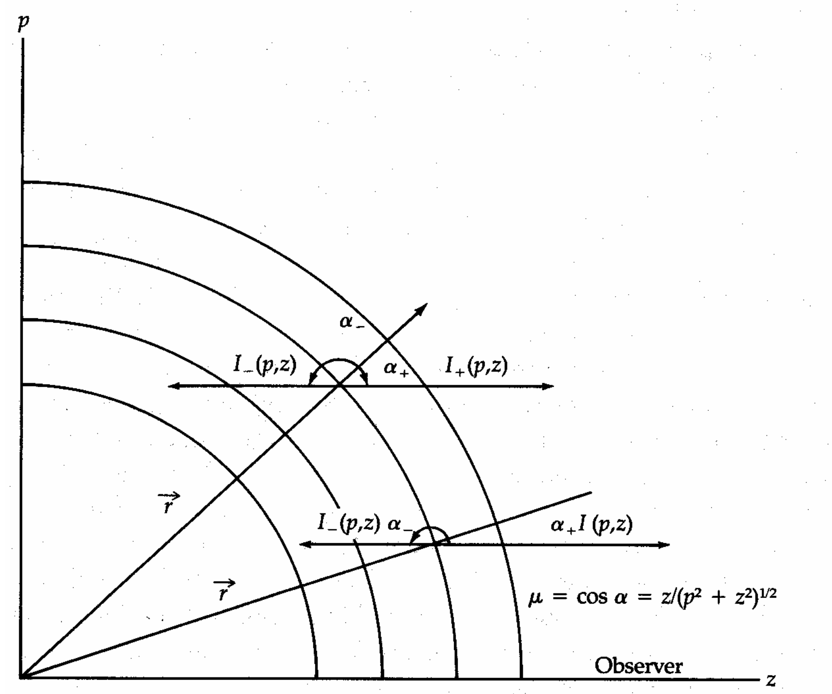

Figure 10.3 describes 'impact space' for spherical transport. The z-axis points at the observer, while the *p* -coordinate is perpendicular to z and plays the role of an impact parameter for the photons directed toward the observer. The angle α denotes the angle between a line parallel to z, directed toward the observer, and a radius vector.

Thus any solution that gives us a complete representation of the specific intensity in the *p*-z plane will give a complete description of the radiation field. We can immediately write the equation of transfer for the special beams directed toward or away from the observer as

$$\pm\frac{\partial I_\pm(p,z)}{\partial z} = \rho(\kappa_\nu + \sigma_\nu)[S(p,z) - I_\pm(p,z)] \tag{10.4.20}$$

where the coordinate transformation from *p*-z coordinates to μ-r coordinates is

$$r = (p^2 + z^2)^{1/2} \qquad |\mu| = \frac{z}{(p^2 + z^2)^{1/2}} \tag{10.4.21}$$

For simplicity we denote

$$k_\nu = \rho(\kappa_\nu + \sigma_\nu) \tag{10.4.22}$$

Equation (10.4.20) is a linear first order equation that has a classical solution of

$$I_-(p, z) = \int_z^{(R^2-p^2)^{1/2}} k_\nu(\xi)S_\nu(\xi)e^{-\tau(p,\zeta,z)}\,d\zeta$$
$$I_+(p, z) = \int_0^z k_\nu(\xi)S_\nu(\xi)e^{-\tau(p,z,\zeta)}\,d\zeta + I_-(p, 0)e^{-\tau(p,z,0)}$$
$$\xi^2 \equiv p^2 + \zeta^2 \qquad \tau(p, a, b) = \int_a^b k(\xi)\,d\zeta \tag{10.4.23}$$

While this is a complicated expression, it can be evaluated numerically as long as one has a representation of the source function. Thus, it is now possible to solve for the entire radiation field and recalculate the variable Eddington factor f_ν. Equation (10.4.15) is then solved again for a new value of J_ν and hence S_ν. The entire procedure is repeated until a self-consistent solution is found. Rather than carry out the admittedly messy numerical integration, Mihalas[3] describes a Feautrier-like method to calculate the intensities directly.

A method proposed by Schmid-Burgk[4] assumes that the source function can be locally represented by a polynomial in the optical depth. This analytic function is then substituted into the formal solution in impact space so that the radiation field can be represented in terms of the undetermined coefficients of the source function's approximating polynomials. The moments of the radiation field can then be generated which depend only on these same coefficients. Thus, if one starts with an initial atmospheric structure and a guess for the source function, one can fit that source function to the local polynomial and thereby determine the approximating coefficients. These, in turn, can be used to generate the moments of the radiation field upon which an improved version of the source function rests. An excellent initial guess for the source function is $S_\nu = B_\nu$, and unless scattering completely dominates the opacity, the iteration process converges very rapidly.

It is clear that the spherical atmosphere poses significant difficulties over and above those found in the plane-parallel atmosphere. However, there are very few differences that are fundamental in nature. All present methods rely on the global symmetry of spherical stars, and it seems likely that those stars with atmospheres sufficiently extended to require the spherical treatment will also be subject to other forces, such as rotation, that further distort the atmospheres so that even this global symmetry is lost. However, such studies can offer insight into the severity of the effects that we can expect from the geometry.

We have only skimmed the surface of the methods and techniques devised to solve the equation of radiative transfer. The methods discussed merely comprise some of the more popular and successful methods currently in use. We have left to the studious reader the entire area of the "exact approximation" and the H-functions of Chandrasekhar[1] (pp. 105 to 126). No mention has been made of invariant embedding and the voluminous literature written for Linear two-point boundary-value problems. Many of these techniques have proved useful in solving specific radiative transfer problems, and those who would count themselves experts in this area should avail themselves of that literature. There is an entire field of study surrounding the transfer of radiation within spectral lines, some of which will be discussed later, but much of which will not be. This material is important for anyone interested in problems requiring line-transfer solutions. However, the methods presented here suffice for providing the solution to half of the task of constructing a normal stellar atmosphere, and next we turn to the solution of the other half of the problem.

Problems

1. Find the general expression for

$$\int_0^\infty \frac{(t-\tau)^n E_1|t-\tau|}{n!}\,dt$$

2. Find the eigenvalues k_α and $L_{+\alpha}$ for the discrete ordinate solution to the semi-infinite plane-parallel gray atmosphere for $n = 8$.

3. Repeat Problem 2 for the double-gauss quadrature scheme for $n = 8$.

4. If there is an arbitrary iterative function $\mathbf{\Phi}(x)$ such that

$$x_{k+1} = \Phi(x_k)$$

then an iterative sequence defined by $\mathbf{\Phi}(x_k)$ will converge to a fixed point x_0 if and only if

$$\left|\frac{\partial\Phi(x)}{\partial x}\right| < 1 \qquad |x_k| \le |x| \le |x_0|$$

Use this theorem to prove that any fixed-point iteration scheme will provide a solution for

$$B(\tau) = \tfrac{1}{2}\int_0^\infty B(t)E_1|t-\tau|\,dt$$

5. Find a general interpolative scheme for $I(\tau,\mu)$ when $\mu < 0$ for the discrete ordinate approximation. The interpolative formula should have the same degree of precision as the quadrature scheme used in the discrete ordinate solution.

6. Consider a pure scattering plane-parallel gray atmosphere of optical depth t_0, illuminated from below by $I(\tau_0,+\mu) = I_0$. Further assume that the surface is not illuminated [that is, $I(0,-\mu) = 0$. Use the Eddington approximation to find $F(\tau)$, $J(\tau)$, and $I(0,+\mu)$ in terms of I_0 and τ_0.

7. Show that in a gray atmosphere

$$\frac{dP_r}{dT} = \frac{16\sigma}{3c}\frac{T^3}{1 + dq(\tau)/d\tau}$$

8. Use the first of the Schwarzschild-Milne integral equations for the source function in a gray atmosphere [equation (10.2.6)] to derive an integral equation for the Hopf function $q(\tau)$.

9. Show that no self-consistent solution to the equation of radiative transfer exists for a pure absorbing plane-parallel gray atmosphere in radiative equilibrium where the source function has the form

$$S(\tau) = a + b\tau$$

10. Show that the equation of transfer in spherical coordinates

$$\cos\theta\frac{\partial I_\nu(r,\theta)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu(r,\theta)}{\partial\theta} = \rho(\kappa_\nu + \sigma_\nu)[S_\nu(r) - I_\nu(r,\theta)]$$

transforms to

$$\pm\frac{\partial I_\pm(p,z)}{\partial z} = \rho(\kappa_\nu + \sigma_\nu)[S_\nu(p,z) - I_\pm(p,z)]$$

in impact space where $r^2 = (p^2+z^2)$, and $|\mu| = z/r$.

11. Derive an integral equation for the mean intensity $J_\nu(\tau_\nu)$ when the source function is given by

$$S_\nu(\tau_\nu) = \kappa_\nu B_\nu(\tau_\nu) + \frac{\sigma_\nu}{2}\int_{-1}^{+1}[1 - (\mu')^2]I_\nu(\mu',\tau_\nu)\,d\mu'$$

12. Numerically obtain a solution for the Schwarzschild-Milne integral equation for the source function in a gray atmosphere by solving equation (10.2.11) for the ratio of the source function at eight points in the atmosphere to its value at one point. Describe why you picked the points as you did, and compare your result with that obtained from the Eddington approximation.

13. Using equation (10.2.21), show that equations (10.2.22) and (10.2.23) follow from the discrete ordinate equation of transfer [equation (10.2.20)].

14. Show that equations (10.2.29) and (10.2.30) follow from the substitution of the solution for the discrete ordinate method [equation (10.2.26)] into the definition for the moments of the radiation field, J(τ), and F.

15. Show that equation (10.2.36) is indeed the eigen-equation for the nonconservative gray atmosphere.

16. Use the Feautrier method to solve the problem of radiative transfer in a gray atmosphere.

References and Supplemental Reading

1. Chandrasekhar, S.: *Radiative Transfer*, Dover, New York, 1960, pp. 54 - 68.

2. Kurucz, R.L.: *ATLAS: A Computer Program for Calculating Model Stellar Atmospheres*, Smithsonian Astrophysical Observatory Special Report 309, 1970.

3. Mihalas, D.: *Stellar Atmospheres*, 2d ed., W.H.Freeman, San Francisco, 1978, pp. 250 - 255.

4. Schmid-Burgk, J.: *Radiative Transfer through Spherical-Symmetric Atmospheres and Shells*, Astron. & Astrophy. 40, 1975, pp. 249 - 255.

Virtually every book about stellar atmospheres provides an introduction to the subject that is worth perusing. Some are more valuable than others in providing insight into the physics of the atmosphere. In the area of radiative transfer, the definitive mathematical treatise is still

Chandrasekhar, S.: *Radiative Transfer*, Dover, New York 1960.

However, students should not try to read this work until they have gained considerable familiarity with the problem. One of the clearest and most comprehensive descriptions of the gray atmosphere and various methods of solution of the radiative transfer problem is found in

Kourganoff, V.: *Basic Methods in Transfer Problems - Radiative Equilibrium and Neutron Diffusion*, Dover, New York, 1963, pp.86 - 125.

An extremely complete discussion of Λ-operators is given in this same reference (pp. 40 - 85). Dimitri Mihalas provides a good description of the gray atmosphere in both editions of his book on stellar atmospheres, but of the two, I prefer the first edition;

> Mihalas, D.:*Stellar Atmospheres*, 1st ed., W.H.Freeman, San Francisco, 1970, pp.34 -66.

For a lucid discussion of the relative merits of solutions to the integral equations of radiative transfer, see

> Kalkofen, W. *A Comparison of Differential and Integral Equations of Radiative Transfer*, J. Quant. Spectrosc. & Rad. Trans. 14, 1974, pp. 309 - 316.

For a general background of the subject as considered by some of the finest minds of the twentieth century, everyone should spend some time eading *Selected Papers on the Transfer of Radiation*, edited by D. H. Menzel (Dover, New York, 1966). All these papers are of landmark quality, but I found this one to be most rewarding and somewhat humbling:

Schuster, A.:*Radiation through a Foggy Atmosphere*, Ap.J. 21, 1905 pp.1 - 22,

It is clear that Arthur Schuster identified and understood most of the important aspects of scattering theory in radiative transfer without the benefit of the work of the rest of the twentieth century that is available to the contemporary student of physics. Much of the work on neutron diffusion theory deals with the same mathematical formalisms that serve radiative transfer theory, and we should be ever mindful of the physics literature on that subject if we are to appreciate the full breadth of the nature of the problems posed by the flow of radiation through the outer layers of stars. Finally, it would be a mistake to ignore the substantial contribution from the Russian school of radiative transfer theory. Perhaps the finest example of their efforts can be found in

Sobolev, V. V.:*A Treatise on Radiative Transfer*,(Trans. S. I. Gaposchkin), Van Nostrand, Princeton, N.J., 1963.

The approaches described in this book are insightful, novel, and particularly useful in dealing with some of the more advanced problems of radiative transfer.