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The Basics of Classical Mechanics

Celestial mechanics is a specialized branch of classical mechanics and a proper understanding of the subject requires that one see how it is embedded in this larger subject. One might describe the fundamental problem of celestial mechanics as the description of the motion of celestial objects that move under the influence of the gravitational forces present in the solar system. The approach of classical mechanics to this problem would be to divide it into two parts. The first of these would be to write the equations of motion for an object moving under the influence of an arbitrary collection of mass points. The second part then consists of solving those equations. Therefore it is appropriate that we spend some time with the fundamentals of classical mechanics so that their relation to the more specific subject of celestial mechanics is clear. The most basic concept of all of theoretical physics is the notion of a conservation law. But before one can discuss the conservation of quantities, one must define them.

3.1 Newton's Laws and the Conservation of Momentum and Energy

Newton's famous laws of motion can largely be taken as a set of definitions. Consider the second law

$$\vec{F} = m\vec{a} = \frac{d(m\vec{v})}{dt} \equiv \frac{d\vec{p}}{dt} . \quad (3.1.1)$$

Here \vec{p} is defined as the linear momentum. It is a simple matter to describe operationally what is meant by mass m and acceleration \vec{a} . However, a clear operational definition of what is meant by force that does not make use of some

form of equation (3.1.1) is more difficult. However, one can use the first law to describe a situation wherein forces are absent, and that is sufficient to arrive at a conservation law for linear momentum, namely

$$\left. \begin{array}{l} \vec{F} = 0 = \frac{d\vec{p}}{dt} \\ \vec{p} = \text{const.} \end{array} \right\} . \quad (3.1.2)$$

We may use the concept of linear momentum to define an additional quantity called angular momentum (\vec{L}) so that

$$\vec{L} \equiv \vec{r} \times \vec{p} . \quad (3.1.3)$$

Since \vec{p} is a conserved quantity, it would seem plausible that angular momentum will also be a conserved quantity.

Let us define a force-like quantity, by analogy with the angular momentum, called the torque as

$$\vec{N} \equiv \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} - \left(\frac{d\vec{r}}{dt} \times \vec{p} \right) . \quad (3.1.4)$$

However,

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0 . \quad (3.1.5)$$

Therefore

$$\vec{N} = \frac{d\vec{L}}{dt} . \quad (3.1.6)$$

Thus a force free situation will also be a torque free situation and the angular momentum will be constant and conserved in exactly the same sense as the linear momentum.

Finally, let us define the concept of work as the line integral of force over some path or

$$W_{a,b} \equiv \int_a^b \vec{F} \cdot d\vec{s} . \quad (3.1.7)$$

Note that the quantity called work is a scalar quantity as only the component of the force directed along the path contributes to the work. We may use the notion

of work to say something about the nature of the forces along the path. Specifically, if no net work is done while completing a closed path so that

$$\oint \vec{F} \cdot d\vec{s} = 0 \quad , \quad (3.1.8)$$

and this is true for *any closed path*, the force is said to be a conservative force. Now there is a theorem in mathematics known as Stokes theorem where

$$\int_c \vec{Q} \cdot d\vec{s} = \int_s (\nabla \times \vec{Q}) \cdot d\vec{A} \quad . \quad (3.1.9)$$

The left hand quantity is a line integral along some curve C which bounds a surface S. The quantity on the right hand side is a surface integral over that surface where $d\vec{A}$ is a unit vector normal to the differential area dA . Applying this theorem to equation (3.1.8) we get

$$\oint_c \vec{F} \cdot d\vec{s} = \int_s (\nabla \times \vec{F}) \cdot d\vec{A} \quad . \quad (3.1.10)$$

But for conservative force fields this result must be true for all paths and hence the right hand side must hold for all enclosed areas. This can only be true if the integrand of the right hand integral is itself zero so that

$$\nabla \times \vec{F} = 0 \quad . \quad (3.1.11)$$

Since the curl of the gradient is always zero (i.e. ∇ points in the same direction as itself), we can write a conservative force as the gradient of some scalar V so that

$$\vec{F} = -\nabla V \quad . \quad (3.1.12)$$

The quantity V is called the potential energy. Since the quantity \vec{F} is related to operationally defined parameters, the potential energy is defined only through its gradient. Thus we may add or subtract any constant to the potential energy without affecting measurable quantities. This is often done for convenience and the additive constant must be included in any self consistent use of the potential. From equations (3.1.7) and (3.1.12) it is clear that we can determine the work done in moving from point a to point b in terms of the potential as

$$W_{a,b} \equiv \int_a^b \vec{F} \cdot d\vec{s} = -\int_a^b \nabla V \cdot d\vec{s} = -\int_a^b \sum_i \frac{\partial V}{\partial x_i} \hat{x}_i \cdot d\vec{s} \quad . \quad (3.1.13)$$

However,

$$d\vec{s} = \sum_j \hat{x}_j dx_j \quad , \quad (3.1.14)$$

so that

$$W_{a,b} = -\int_a^b dV = V(a) - V(b) \quad . \quad (3.1.15)$$

Thus the amount of work done on an object is simply equal to the change in the potential energy in going from a to b. Now we may write

$$\int_a^b \vec{F} \bullet d\vec{s} = \int_a^b \frac{d\vec{p}}{dt} \bullet d\vec{s} = \int_a^b m \frac{d\vec{v}}{dt} \bullet \vec{v} dt = \frac{1}{2} m \int_a^b v^2 dt \quad . \quad (3.1.16)$$

so that the change in the kinetic energy of the particle in going from a to b is

$$T(a) - T(b) = V(a) - V(b) \quad . \quad (3.1.17)$$

Thus the sum of the kinetic and potential energies E is the same at points a and b so that

$$T(b) + V(b) = T(a) + V(a) \quad . \quad (3.1.18)$$

This is nothing more than a statement of the conservation of energy. Clearly energy conservation is a weaker conservation law than conservation of momentum as we had to assume that the force field was conservative in order to obtain it.

3.2 Virtual Work, D'Alembert's Principle, and Lagrange's Equations of Motion

Consider a system of particles that are not subject to any constraints. A constraint is something that cannot be represented *in a general way* by the forces acting on all the objects. For example, an object moving under the influence of gravity but constrained to roll on the surface of a sphere for part of its motion would be said to be subject to constraints imposed by the sphere. Such constraints imply that forces are acting on the object in such a way as to constrain the motion, but the forces are not known *a priori* and must be found as part of the problem. Only their effect on the object (i.e. its constrained motion) is known. Some

constraints can be expressed in terms of the coordinates of the problem and are known as *holonomic constraints*. Constraints that cannot be written in terms of the coordinates alone are called *nonholonomic constraints*. The rolling motion of an object where there is no slippage is an example. The constraint here is on the velocity of the point in contact with the surface. We will leave the consideration of such systems for an advanced mechanics course.

The notion of virtual work is a creation of James Bernoulli and comes about from considering infinitesimal displacements of the particles that are subject to forces \vec{F}_i . We can call these displacements $\delta\vec{r}_i$. If these infinitesimal displacements can be called virtual displacements, then $\vec{F}_i \cdot \delta\vec{r}_i$ can be called the virtual work done on the i th particle. The virtual nature of these displacements becomes clear when we require that the forces \vec{F}_i do not change in response to the virtual displacements $\delta\vec{r}_i$ in contrast to the case for real displacements. For a system in equilibrium, the forces on the individual particles vanish and therefore so does the virtual work. For a dynamical system subject to Newton's laws of motion we can say that the forces are balanced by the accelerative response of the system so that

$$\sum_i (\vec{F} \cdot \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \quad . \quad (3.2.1)$$

This is known as D'Alembert's principle and is useful for what we can derive from it.

It may be the case that the \vec{r}_i 's are not all linearly independent. To this point, the choice of the coordinate system used to represent the motion of the system has been arbitrary. Thus it is entirely possible that the coordinates of choice will not be independent of one another. Holonomic constraints can also produce a set of coordinates that are not linearly independent. However, we can hardly expect to unravel the dynamical motion of a system of particles if the coordinates chosen to represent them depend on each other. Therefore, we shall require that the coordinates chosen to represent the system are indeed linearly independent. We shall call any set of coordinates that are linearly independent and describe all the particles of the system *generalized coordinates*. Now consider a transformation from our initial arbitrary set of coordinates to a set of coordinates which are linearly independent and which we shall denote by q_i . We could write such a transformation as

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j} dq_j \quad . \quad (3.2.2)$$

However, since the q_j 's are linearly independent we get

$$\delta q_j = \sum_k \frac{\partial q_j}{\partial q_k} dq_k = \sum_k \delta_{jk} dq_k = dq_j \quad . \quad (3.2.3)$$

Substitution of this into D'Alembert's principle gives

$$\left. \begin{aligned} \sum_j \left[\sum_i \bar{\mathbf{F}}_i \cdot \frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \right] \delta q_j - \sum_i m_i \frac{d^2 \bar{\mathbf{r}}_i}{dt^2} \cdot \sum_j \frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \delta q_j &= \\ \sum_j Q_j \delta q_j - \sum_j \sum_i m_i \frac{d}{dt} \left(\frac{d \bar{\mathbf{r}}_i}{dt} \cdot \frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} - \frac{d \bar{\mathbf{r}}_i}{dt} \cdot \left[\frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \right] \right) \delta q_j & \end{aligned} \right\} . (3.2.4)$$

Here we have used Q_j to stand for the term in parentheses on the left hand side of the equation. Now since the velocity of any particle can be written in terms of the generalized coordinates as

$$\bar{\mathbf{v}}_i = \sum_j \frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial \bar{\mathbf{r}}_i}{\partial t} \quad , \quad (3.2.5)$$

we may calculate its partial derivative with respect to the *total* time derivative of the generalized coordinates as

$$\frac{\partial \bar{\mathbf{v}}_i}{\partial \dot{q}_k} = \sum_k \left(\frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \right) \left(\frac{\partial q_j}{\partial \dot{q}_k} \right) \left(\frac{dq_j}{dt} \right) = \sum_j \left(\frac{\partial \bar{\mathbf{r}}_i}{\partial q_j} \right) \left(\frac{\partial \dot{q}_j}{\partial \dot{q}_k} \right) = \frac{\partial \bar{\mathbf{r}}_i}{\partial q_k} \quad . \quad (3.2.6)$$

This allows us to write the expanded form of D'Alembert's principle as

$$\left. \begin{aligned} \sum_i (\bar{\mathbf{F}}_i - \dot{\bar{\mathbf{p}}}_i) \cdot \delta \bar{\mathbf{r}}_i &= 0 \\ &= \sum_j Q_j \delta q_j - \sum_i \sum_j \left[\frac{d}{dt} \left(m_i \bar{\mathbf{v}}_i \cdot \frac{\partial \bar{\mathbf{v}}_i}{\partial \dot{q}_i} \right) - \left(m_i \bar{\mathbf{v}}_i \cdot \frac{\partial \bar{\mathbf{v}}_i}{\partial q_j} \right) \right] \delta q_j \end{aligned} \right\} . \quad (3.2.7)$$

From the definition of kinetic energy, we get

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 \quad . \quad (3.2.8)$$

However, this result must be true for arbitrary virtual displacements of the generalized coordinates δq_j . Such can only be the case if it is true for each term of the sum. Therefore

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad . \quad (3.2.9)$$

From the definition of Q_j [see equation (3.2.4)] we can write for conservative forces that

$$Q_j = \sum_i \vec{F} \cdot \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = - \sum_i \nabla V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = - \sum_i \left(\frac{\partial V}{\partial \vec{r}_i} \right) \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = - \frac{\partial V}{\partial q_j} \quad . \quad (3.2.10)$$

Very rarely is the potential energy an *explicit* function of time so that

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \quad . \quad (3.2.11)$$

From equations (3.2.10) and equation (3.2.11) it is clear that we can combine the potential energy with the kinetic energy in equation (3.2.9) and thereby eliminate the Q_j s. So define

$$\mathcal{L} \equiv T - V \quad . \quad (3.2.12)$$

which is known as the Lagrangian. In terms of the Lagrangian, equation (3.2.9) becomes

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \left(\frac{\partial \mathcal{L}}{\partial q_j} \right) = 0 \quad . \quad (3.2.13)$$

These are known as Lagrange's equations of motion and their solution constitutes the solution of the first part of the basic problem of classical mechanics. The utility of Lagrange's equations of motion is clear. Given a set of coordinates that are linearly independent, find an expression for the kinetic and potential energies in terms of those coordinates and their time derivatives. Equation (3.2.13) then provides a mechanical means for generating the equations of motion for the particle of interest in the chosen coordinates.

Let us consider as an example the equations of motion for two point masses moving under the influence of their mutual self-gravity. For a generalized set of coordinates, let us use Cartesian coordinates with the origin at the center of mass of the system. Further let the mass of one particle be m and the total mass of the system be M . Since this is an isolated system, the motion of the center of mass can be taken to be zero. We can always transform to an inertial frame that moves with the center of mass. Thus the kinetic and potential energies can be written as

$$\left. \begin{aligned} T &= \frac{1}{2} m v^2 + \frac{1}{2} M^2 (0) = \frac{1}{2} \sum_j \left(\frac{dx_j}{dt} \right)^2 \\ V &= \frac{-GmM}{\left(\sum_j (x_j^2) \right)^{\frac{1}{2}}} \end{aligned} \right\} , \quad (3.2.14)$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \sum_j \left(\frac{dx_j}{dt} \right)^2 + \frac{GmM}{\left(\sum_j (x_j^2) \right)^{1/2}} . \quad (3.2.15)$$

Substituting this into equation (3.2.13) and remembering that q_i is x_i we can write the equations of motion as

$$m \frac{d^2 x_i}{dt^2} + \frac{GmM x_i}{\left(\sum_j x_j^2 \right)^{3/2}} = 0 . \quad (3.2.16)$$

Before turning to the problem of determining the potential for an arbitrary collection of mass points we will briefly discuss a related method of obtaining the equations of motion.

3.3 The Hamiltonian

There is an approach to mechanics due to Sir W. R. Hamilton that is very similar to Lagrange's and has wide ranging applications in theoretical physics. The Hamiltonian formulation adds nothing new in the form of physical laws, but provides what many feel is a much more powerful formalism with which any student of the physical sciences should be familiar.

The basic idea of the Hamiltonian formulation is to write equations of motion in terms of coordinates and the momentum instead of the coordinates and their time derivatives. Lagrange's equations of motion are second order differential equations requiring $6N$ constants of integration, which are usually the initial values of \dot{q}_i and q_i . If we choose as generalized coordinates of the problem (q_i, p_i, t) , then we can in principle write $2N$ first order equations of the motion still requiring $6N$ constants, but reducing the order of the equations to be solved. This can be accomplished by subjecting the Lagrangian equations of motion to a transformation known as the Legendre transformation. First, let us *define* what we will mean by the generalized momenta as

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} . \quad (3.3.1)$$

Lagrange's equations and this definition allow us to write

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} . \quad (3.3.2)$$

Now let us just guess a function of the form

$$H(p, q, t) = \sum_i \dot{q}_i p_i - \mathcal{L}(q, \dot{q}, t) , \quad (3.3.3)$$

and call it the Hamiltonian. The total differential of the right hand side of equation (3.3.3) yields

$$dH = \sum_i \dot{q}_i dp_i + \sum_i p_i dq_i - \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) d\dot{q}_i - \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} \right) dq_i - \left(\frac{\partial \mathcal{L}}{\partial t} \right) dt . \quad (3.3.4)$$

Now the second and third terms cancel by virtue of the definition of the generalized momenta [equation (3.3.1) multiplied by $d\dot{q}_i$ and summed over i]. The partial derivative of the Lagrangian in the fourth term can be replaced by equation (3.3.2) so that the total differential of H becomes

$$\left. \begin{aligned}
dH &= \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \left(\frac{\partial \mathcal{L}}{\partial t} \right) dt \\
&= \sum_i \left(\frac{\partial H}{\partial q_i} \right) dq_i + \sum_i \left(\frac{\partial H}{\partial p_i} \right) dp_i + \frac{\partial H}{\partial t} dt
\end{aligned} \right\} . \quad (3.3.5)$$

The far right hand part of equation (3.3.5) is just the definition of a total differential. Since the generalized coordinates are linearly independent, the three terms on each side of equation (3.3.5) must be separately equal so that

$$\left. \begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= - \frac{\partial H}{\partial q_i} \\
\frac{\partial \mathcal{L}}{\partial t} &= - \frac{\partial H}{\partial t}
\end{aligned} \right\} . \quad (3.3.6)$$

These equations are known as the *canonical equations of Hamilton* and they form a set of $2n$ first order equations for the motion of the constituents of the system.

From the definition of a total time derivative we can write

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) + \frac{\partial H}{\partial t} . \quad (3.3.7)$$

However, by substituting for \dot{q}_i and \dot{p}_i from the Hamilton equations of motion, we see that the term in parentheses vanishes so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = - \frac{\partial \mathcal{L}}{\partial t} . \quad (3.3.8)$$

Thus if the Lagrangian is not an *explicit* function of time, then the Hamiltonian will not vary with time at all and will therefore be a constant of the motion. If, in addition, the transformation to the generalized coordinates also does not depend explicitly on time, the Hamiltonian will be the total energy of the system. In most celestial mechanics problems this is indeed the case. The

potential depends only on position and not explicitly on time and the generalized coordinates are usually the position coordinates themselves. Thus the Hamiltonian is a constant of the system and is equal to the total energy. The primary exception to this is when analysis is done in a rotating or non-inertial coordinate frame. Then the transformation to the generalized coordinates does explicitly involve time and thus the Hamiltonian is not the total energy of the system. However, if the Lagrangian is not an explicit function of time, the Hamiltonian is still a constant of the motion.

One standard way of proceeding with a classical mechanics problem is to find the Lagrangian by determining the potential. Then equation (3.3.3) can be used directly to calculate the Hamiltonian. Then equations (3.3.6) yield the equations of motion and usually one of the constants of the motion has been found in the process. This formalism is so powerful that it forms the basis for a great deal of quantum mechanics. It is clear that for celestial mechanics, the central remaining problem to finding the equations of motion is the determination of the potential and so in the next chapter we will turn to various methods by which that can be done.

Chapter 3: Exercises

1. The escape velocity from the Earth is the minimum velocity required to escape the influence of the Earth's gravitational field. Neglecting atmospheric drag, use basic conservation laws to find the value for the escape velocity from the surface of the Earth.
2.
 - a: Find the equations of motion for a rocket projected vertically from the surface of the Earth. Again, neglect atmospheric drag.
 - b: Assuming the rate of mass loss from the rocket is constant and equal to $1/60$ of the initial mass per sec, show that if the exhaust velocity is 2073 m/s, then for the rocket to reach escape velocity, the ratio of the mass of the fuel to the empty rocket must be about 300.
3. Consider a system of n particles moving under the influence of gravity alone.
 - a: Write down the Lagrangian for the system.
 - b: Find the equations of motion in Cartesian coordinates for the system
4. Consider a single particle moving under the influence of the potential Φ where

$$\Phi = \sum_{k=0}^n \frac{a_k \cos(k\theta)}{(k+1)} .$$

Find the Lagrangian, Hamiltonian, and the equations of motion in spherical coordinates for the particle.