7

The Observational Determination of Orbits

In the last chapter we saw how to find the position on the sky of a object given the parameters that describe the orbit of the object. That is about half of the fundamental problem of celestial mechanics. The other half is the reverse. Namely, given some observational information about the motion of the object, one would like to determine the orbital elements that specify the motion. This, and Chapter 6, enable one to predict the future location and motion of the object. These two parts of the description of orbital motion constitute the solution of the primary problem of celestial mechanics.

It is clear from what we have done in Chapters 4 and 5 that the solution of the equations of motion for n-bodies requires 6n constants of integration. For two bodies half of the constants are involved in describing the motion of the center of mass, while the remaining six specify the location of a particle in its orbit and the orientation of that orbit with respect to a specified coordinate system. Thus, for objects in orbit about the sun we have only the six orbital elements that represent the six linearly independent constants required for the solution of the equations of motion. In order to determine these six linearly independent orbital elements, we will need six linearly independent pieces of information. There are many different forms that this information may have. For example, one might have the position and velocity at some instant in time. These two vectors clearly provide six independent pieces of information as they constitute the classical initial values for the integration of the Newtonian equations of motion. However, they are not the quantities traditionally available to the astronomer. Classically, one observes the position of an object as seen projected against the celestial sphere. Such an observation is comprised of two angular coordinates and the time of observation.

This represents two linearly independent pieces of information so that one would need three such observations in order to determine the orbital elements. In principle, one might also be able to measure the radial velocity with respect to the earth, but this is only one additional independent piece of information. Thus, if one had two positions and the radial velocities of the object at those positions, the problem would be determined.

In practice, all observations are subject to error and this will be reflected in errors in the orbital elements. Therefore, the accurate determination of orbital elements will make use of a large number of observations combined in such a way as to reduce the resultant error of the final result. The combination of the observations usually employs some principle such as Legendre's principle of least squares or more contemporarily, the related maximum likelihood principle. However, all of these methods require the relationship between the orbital elements to be determined and the particular type of observations to be specified. Since this relationship is, in general, nonlinear, we shall consider several different and specific cases. As an example and for traditional reasons, we shall consider the problem of determining the orbital elements for an object in orbit about the sun. However, the approaches are much more general and are applicable for determining the orbits of objects revolving about most any object where the potential is that of a point mass.

7.1 Newtonian Initial Conditions

In Chapters 3 and 5 we found that the two body problem will have two integrals of the motion, the angular momentum and the total energy. Integrals of the motion are useful for our purpose since they are indeed constant for all parts of the orbit and therefore apply as constants for all possible observations. They represent constraints that all observations must satisfy, and they can be directly related to the orbital elements. Therefore we will begin by discussing what they can tell us about positions and velocities and vice versa. Let us assume that we know a position and velocity at some instant in time. This is essentially the initial value information that would be needed for the direct solution of the Newtonian equations of motion. The definition of angular momentum requires that

$$\vec{r} \times \dot{\vec{r}} = \vec{L}/m = |r||v| \sin\theta \, \hat{\ell} \quad , \tag{7.1.1}$$

and the angular momentum is an integral of the motion. From the solution of the two body problem [see equations (6.2.12), and (6.2.14)] and the properties of an ellipse we know that

$$P = L^2 / GMm^2 = a(1 - e^2) \quad . \tag{7.1.2}$$

If we combine this with the expression for the velocity of an object moving in a central force field [i.e. equation (5.4.11)], we get

$$\dot{\vec{r}} \bullet \dot{\vec{r}} = v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right] \quad . \tag{7.1.3}$$

This is often called the energy integral since it is basically derived from the conservation of energy. Older books on celestial mechanics refer to it by the old Latin name *vis viva Integral.* It immediately supplies us with a value for the semi-major axis.

$$a = GMr / (2GM - rv^2) \quad , \tag{7.1.4}$$

which is one of the orbital elements we seek.

Now we may obtain the value of the orbital eccentricity e by using equation (7.1.1) to replace the value of (L/m) in equation (7.1.2) and obtain

$$e^2 = 1 - [(rv\sin\theta)^2 / GMa] = 1 - [2 - (r/a)](r/a)\sin^2\theta \quad . \tag{7.1.5}$$

With the semi-major axis, a, and the orbital eccentricity, e, we can turn directly to the equation for the orbital ellipse (6.2.14) to obtain the cosine of the true anomaly as

$$e\cos\nu = (a/r)(1 - e^2) - 1 = (e^2 - \cos^2\theta)^{1/2}\sin\theta - \cos^2\theta \quad . \tag{7.1.6}$$

The right hand side of equation (7.1.6) is obtained with the aid of equation (7.1.7). The proper quadrant for ν may be found from the sign of the radial velocity, $\dot{r} = v\cos\nu$, which we get by differentiating equation (6.2.14) with respect to time, noting that $\dot{\nu}$ can be obtained from the areal velocity [see equations (5.2.2, 3)] as L/mr^2, and that P may be eliminated with the aid of equations (7.1.1) and (7.1.2) so that

$$\left.\begin{aligned} e\sin\nu &= (Pmv\cos\theta / L) = (a/r)(1 - e^2)\cot\theta \\ &= \cos\theta\{\sin\theta + [\sin^2\theta - (1 - e^2)\cos^2\theta]^{1/2}\} \end{aligned}\right\} . \tag{7.1.7}$$

The true anomaly ν and eccentricity, e, allow us to directly calculate the eccentric anomaly (E) from equation (6.2.27) and, by means of Kepler's Equation [equation (6.2.25)], the mean anomaly (M). The mean anomaly, in turn, allows the calculation of the time of perihelion passage since the mean daily motion (n) depends only on the period which, in turn, depends only on the semi-major axis so that

$$T_0 = t_1 - (E - e\sin E)/n = t_1 - (a/GM)^{1/2} a(E - e\sin E) \quad . \tag{7.1.8}$$

Here t_l refers to the time at which the observations of the position and velocity are made. Thus equations (7.1.4), (7.1.7), and (7.1.8) determine the shape of the orbit and the orbital element that locates the object in its orbit. The information that has been used to determine these orbital elements is just the magnitude of the angular momentum and energy and the angle between the position and velocity vector. These are three linearly independent pieces of information and they determine three orbital elements. Clearly the energy and angular momentum determine the shape and size of the orbit as they are integrals of the motion and are constants for all points in the orbit. Taken together with the angle between the position and velocity vectors, they are sufficient to locate the particle in that orbit.

The remaining three orbital elements specify the orientation of the orbit and must be determined from information uniquely related to its orientation. The angular momentum vector always points normal to the orbit and, being an integral of the motion, is sufficient to specify the orbit's orientation. A unit vector pointing in the direction of the angular momentum vector contains all the information necessary to specify the orbital orientation. It can be specified in terms of the position. and velocity vectors as

$$\hat{n} = \frac{\vec{L}}{L} = \frac{(\vec{r} \times \dot{\vec{r}})}{|\vec{r}||\dot{\vec{r}}|\sin\theta} \quad . \tag{7.1.9}$$

Thus, the components of that vector in any particular coordinate system will specify the orientation of the orbit in that coordinate system. The components in the ecliptic coordinate system, yield two of the remaining three orbital elements from

$$\left.\begin{aligned} n_x &= \sin\Omega\sin(i) \\ n_y &= -\cos\Omega\sin(i) \\ n_z &= \cos(i) \end{aligned}\right\} \quad . \tag{7.1.10}$$

The remaining orbital elements can be determined by considering a unit vector $(\hat{\eta})$ pointing toward the ascending node and its scalar and vector products with the position vector r which are

$$\left.\begin{aligned} \hat{\eta} \bullet \vec{r} &= r\cos(\nu + \omega) = r_x \cos\Omega + r_y \sin\Omega \\ \hat{\eta} \times \vec{r} &= r\sin(\nu + \omega)\hat{n} \end{aligned}\right\} \quad . \tag{7.1.11}$$

The x-component of the vector cross product is

$$(\hat{\eta}\times\vec{r})_x = r_z\sin\Omega = r\sin\Omega\sin(i)\sin(\nu+\omega) \quad , \tag{7.1.12}$$

which along with the scalar product yields

$$\left.\begin{aligned} \sin(\nu+\omega) &= \frac{r_z}{r\sin(i)} \\ \cos(\nu+\omega)) &= r_x\cos\Omega + r_y\sin\Omega \end{aligned}\right\} \quad . \tag{7.1.13}$$

These two equations are sufficient to unambiguously determine $(\nu + \omega)$ and hence the last remaining orbital element, the argument of perihelion ω.

Thus we have seen how, given what amount to initial conditions of the motion, $[\vec{v}(t_1),\vec{r}(t_1)]$ can be used to determine the orbital elements. It is important to recognize the type of information available and which orbital elements are constrained by that information. Magnitudes of position and velocity vectors specify the magnitudes of the orbital energy and angular momentum. Since these are integrals of the motion, they will determine the size and shape of the orbit. The constancy of the angular momentum vector in space will essentially determine the orientation of the orbit. A combination of both is required to locate the object is its orbit. All methods of determining orbital elements will utilize the observed information in this way. While astronomers rarely are able to determine position and velocity vectors at a given instant, most methods of orbit determination rely on estimating this information from the information that is available.

7.2 Determination of Orbital Parameters from Angular positions Alone

The traditional problem of celestial mechanics involves the determination of the orbital elements given the angular position on the celestial sphere at various times. This information takes the form of pairs of celestial coordinates in some known coordinate system. Since there are six constants of the motion, we will need at least six independent observational constraints or three observations of coordinate pairs. To understand conceptually how this can work, let us consider a method that dates back at least to Johannes Kepler.

a. The Geometrical Method of Kepler

This method determines the planetary orbit with respect to the earth's orbit. In away this is true for all methods since the scale of the solar system is set by the value of the astronomical unit which is generally assumed to be known. However, it is interesting that this method makes no use of physics and only assumes that both the earth and planet are in orbit about the sun. Indeed, this is the method by which Kepler discovered his laws of motion. One begins by determining the sidereal period of the planet, the time required for the planet to return to the same point with respect to the stars as seen in an inertial frame. This is done by measuring the synodic period directly. The synodic period is simply the length of time required for the planet to return to the same place in the sky *as seen from the earth* (see Figure 7.1).

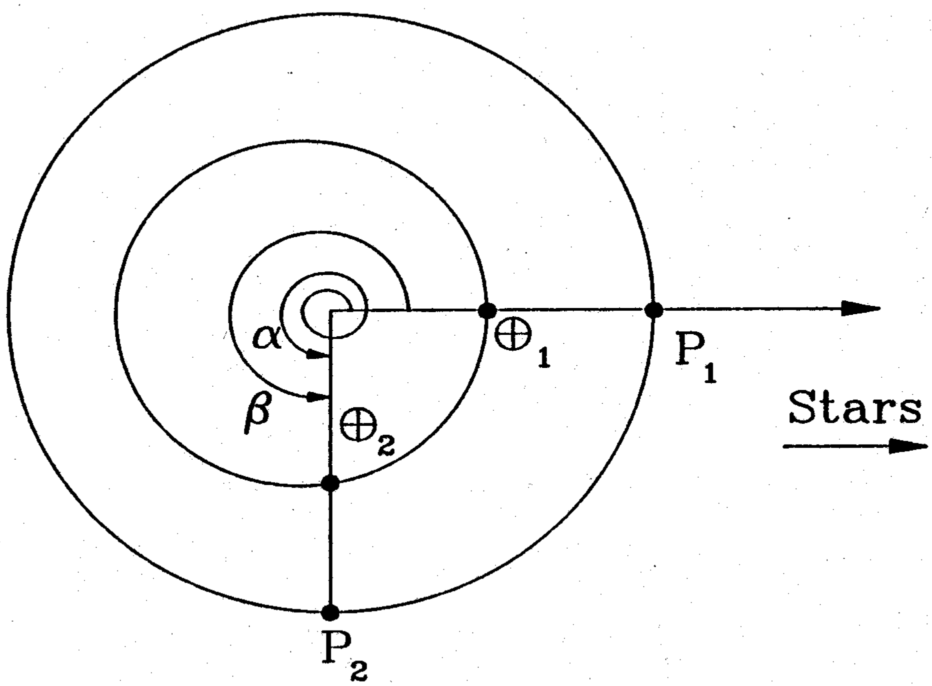

Figure 7.1 shows the orbital motion of a planet and the earth moving from an initial position with respect to the sun (opposition) to a position that repeats the initial alignment. This associated time interval is known as the synodic period of planet p with respect to the earth. The concept of a synodic period need not be limited to the earth and another planet, but may involve any two planets.

Let this period of time be P_s. Now the angular distance traveled by the earth during this time will just be $(2\pi/P_\oplus)\times P_s$ where $P_\oplus$ is the sidereal period of the earth. During the same interval of time the planet will have traveled an angular distance $(2\pi/P_p)\times P_s$. However, since the planets have returned to the same relative position in the sky with respect to the sun, the angular difference in the distance traveled must be 2π. Therefore

$$\left|\frac{2\pi}{P_\oplus}-\frac{2\pi}{P_p}\right|=\frac{2\pi}{P_s} \quad . \tag{7.2.1}$$

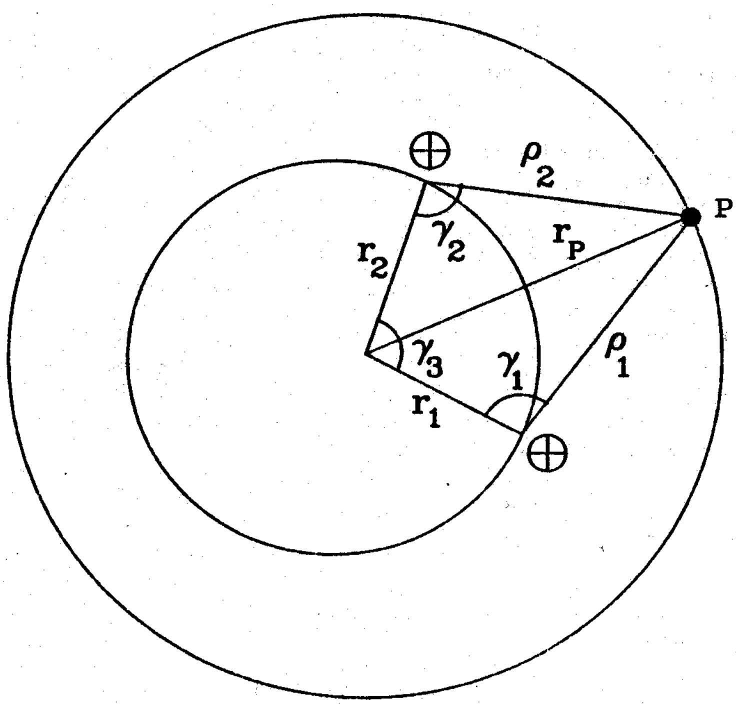

Figure 7.2 shows the position of the earth at the beginning and end of one sidereal period of planet p. If we assume that the distance of the earth to the sun as well as the three angles γ_i are known at each position, then the determination of the remaining parts of the quadrilateral, including the distance to the planet, is a matter of plane trigonometry.

Thus careful observation of the synodic period will lead to the determination of the sidereal period of the planet. While it is true that elliptic orbits will cause difficulties with this approach, it is possible to wait a number of synodic periods until the planet returns arbitrarily close to a given position in the sky and then the method will give the correct result in spite of the orbital eccentricity. While this is not strictly an angular position, it is the measurement of a single item, the synodic period, which then specifies the sidereal period.

Now simply observe the position of the planet at the beginning and end of one sidereal period. It will be seen against the stellar field from two different vantage points as the *sidereal* period of the planet will not in general be commensurate with that of the earth. Thus the planet will lie at the vertex of a quadrilateral formed by the planet, sun and two positions of the earth (see Figure 7.2). Assuming that the orbit of the earth is known, then two sides and three angles of the quadrilateral are known. This enables the remaining sides and diagonals to be determined. If this procedure is carried out throughout the entire orbit of the planet, its entire orbit with respect to the earth can be measured. If the detailed shape of the orbit is known, then clearly the orbital elements that describe the orbit are specified. Much more than the minimum three pairs of observations have gone into this determination, but much less has been assumed. The two-body orbital mechanics that gives rise to the six constants of motion and even allows us to say what minimum amount of information is necessary has not even been used. Let us now consider a method that integrates Newtonian mechanics into the geometrical approach of Kepler.

b. The Method of Laplace

The basic approach of Laplace was to write the equations of motion in terms of the change of a vector from the earth to the object and then to separate the vector into its magnitude and its direction cosines. It is the changes in these direction cosines that essentially constitute the angular measurements that determine the orbital elements. The entire procedure estimates the values for the position and velocity vectors at some instant in time. One then can use the procedure in Section 7.1 to get the orbital elements. This is the schematic procedure that we will follow, but to begin we shall make the following definitions for the vectors involved in the development:

$$\left.\begin{array}{ll} \vec{r}_p & \equiv \textit{the heliocentric radius vector to the planet} \\ \vec{r}_\oplus & \equiv \textit{the heliocentric radius vector to the earth} \\ \vec{\rho} & \equiv \textit{a vector from the earth to the planet} \end{array}\right\} \quad . \qquad (7.2.2)$$

Now let us represent the components of the vector $\vec{p}$ from the earth to the object by their direction cosines specified in terms of the equatorial coordinates of the object so that

$$\vec{\rho} = \rho \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} = \rho\vec{\lambda} = \rho\hat{\rho} \quad . \tag{7.2.3}$$

The vector $\vec{\lambda}$ is simply a unit vector pointing from the earth to the object. I have deliberately continued to use the older notation for the geocentric distance ρ rather than the currently accepted symbol Δ as the latter has too widely an accepted interpretation as the finite difference operator.

The radial equations of motion for both the object and the earth are:

$$\left. \begin{aligned} \frac{d^2\vec{r}_p}{dt^2} &= -\frac{k\vec{r}_p}{r_p^3} \\ \frac{d^2\vec{r}_\oplus}{dt^2} &= -\frac{k\vec{r}_\oplus}{r_\oplus^3} \end{aligned} \right\} \quad , \tag{7.2.4}$$

where

$$\vec{r}_p = \vec{\rho} + \vec{r}_\oplus \quad . \tag{7.2.5}$$

If we use this to eliminate $\vec{r}_p$ from the first equation of motion we get

$$\frac{d^2\vec{\rho}}{dt^2} + \frac{d^2\vec{r}_\oplus}{dt^2} = \frac{d^2\vec{\rho}}{dt^2} - \frac{k\vec{r}_\oplus}{r_\oplus^3} = -\frac{k}{r_p^3}\left(\vec{\rho} + \vec{r}_\oplus\right) \quad , \tag{7.2.6}$$

and using this result we can eliminate the earth's acceleration from its equation of motion and arrive at

$$\frac{d^2\vec{\rho}}{dt^2} + \frac{k\vec{\rho}}{r_p^3} = k\vec{r}_\oplus\left[\frac{1}{r_\oplus^3} - \frac{1}{r_p^3}\right] \quad . \tag{7.2.7}$$

Explicitly differentiating the vector $\vec{\rho}$ we get

$$\frac{d^2\vec{\rho}}{dt^2} = \frac{d^2(\rho\vec{\lambda})}{dt^2} = \ddot{\rho}\vec{\lambda} + 2\dot{\rho}\dot{\vec{\lambda}} + \rho\ddot{\vec{\lambda}} \quad , \tag{7.2.8}$$

which when substituted into equation (7.2.7) yields

$$\ddot{\rho}\vec{\lambda} + 2\dot{\rho}\dot{\vec{\lambda}} + \rho\ddot{\vec{\lambda}} = -\frac{k\vec{\rho}}{r_p^3} + k\vec{r}_\oplus\left[\frac{1}{r_\oplus^3} - \frac{1}{r_p^3}\right] \quad . \tag{7.2.9}$$

Regrouping the terms so that the time derivatives of ρ are collected we get

$$\ddot{\rho}\vec{\lambda} + 2\dot{\rho}\dot{\vec{\lambda}} + \rho\left[\ddot{\vec{\lambda}} + \frac{k\vec{\lambda}}{r_p^3}\right] = k\vec{r}_\oplus\left[\frac{1}{r_\oplus^3} - \frac{1}{r_p^3}\right] \quad . \tag{7.2.10}$$

Except for these time derivatives and $\vec{r}_p$, all the parameters of equation (7.2.10) are known. Remember this is a vector equation so that it constitutes three scalar equations for $\ddot{\rho}, \dot{\rho}$ and ρ. The right hand side involves k and the heliocentric radius vector to the earth $\vec{r}_\oplus$, which is presumed to be known for all the times of the observations. The parameter r_p may be expressed in terms of ρ, $r_\oplus$, and the angle ψ from the law of cosines as

$$r_p^2 = \rho^2 + r_\oplus^2 - 2\rho r_\oplus \cos\psi \quad . \tag{7.2.11}$$

However, this angle can be obtained from the scalar product of $\vec{r}_p$ and $\vec{\lambda}$ as

$$\cos\psi = (\vec{r}_\oplus \bullet \hat{\rho})/|r_\oplus| = (\vec{r}_\oplus \bullet \vec{\lambda})/|r_\oplus| \quad . \tag{7.2.12}$$

Thus, equations (7.2.10-7.2.12) form a closed system of equations for ρ and its first two time derivatives. This must be solved numerically and by iteration due to the nonlinearity of equations (7.2.11), and (7.2.12). Of course the solution depends on having values of $\vec{\lambda}$ and its time derivatives.

For these time derivatives we turn to the observations. Each positional observation consists of a pair of angular coordinates (α,δ) at some particular time t_i. These angular coordinates are sufficient to generate all the components of $\vec{\lambda}$ from equation (7.2.3). Thus three temporal measurements provide three values of the vector $\vec{\lambda}$. Now expand this vector in a Taylor series in time about the first observation so that

$$\vec{\lambda}(t) = \vec{\lambda}(0) + \dot{\vec{\lambda}}(0)t + \tfrac{1}{2}\ddot{\vec{\lambda}}(0)t^2 + \cdots + \quad . \tag{7.2.13}$$

Thus, for the three successive times of observations we can write

$$\vec{\lambda}(t_i) = \vec{\lambda}(0) + \dot{\vec{\lambda}}(0)t_i + \tfrac{1}{2}\ddot{\vec{\lambda}}(0)t_i^2 + \cdots + \quad , \quad i = 1,\cdots,3 \qquad . \qquad (7.2.14)$$

These constitute three linear algebraic vector equations in $\vec{\lambda}$ and its time derivatives. Their solution need only be done once per problem as they provide the constants necessary for the iterative solution of equations (7.2.10 - 7.2.12) for ρ and its time derivatives. However, the solution of these equations is where most of the error in the final solution arises. If the observations are taken too close together, then their linear independence becomes weak and their values (particularly for the second time derivative) small to indeterminate. Simply, too small a section of the orbit is sampled to provide an accurate determination of the orbital elements. If they are taken too far apart in time, then the validity of the Taylor series becomes suspect. In practice, one would use a number of observations and perhaps a longer Taylor series to ensure that the first three terms were accurately determined. Having assured the accurate determination of $\vec{\lambda}$ and its derivatives one can turn to the solution of equations (7.2.10-7.2.12) and obtain values for ρ and its time derivatives. These and $\vec{\lambda}$ determine the position vector for the object in heliocentric coordinates and its time derivative yields the velocity vector for the object, all at the time of the first observation so that

$$\left.\begin{aligned} \vec{r}_p &= \rho\vec{\lambda} + \vec{r}_\oplus \\ \dot{\vec{r}}_p &= \vec{v}_p = \rho\dot{\vec{\lambda}} + \dot{\rho}\vec{\lambda} + \dot{\vec{r}}_\oplus \end{aligned}\right\} \quad . \qquad (7.2.15)$$

We may now use the methods described in the previous section to find the actual orbital elements. Let us turn to a rather more elegant method that avoids many of the problems of the method of Laplace.

c. The Method of Gauss

While the method of determining orbital elements devised by Laplace is conceptually straightforward, it tends to produce poor initial orbital elements. The reason for this lies in the approximation for the temporal behavior of the radius vector $\vec{\rho}$ from the earth to the object. The Taylor series approximation used to obtain derivatives of $\vec{\rho}$ will generally give uncertain values for those derivatives, which, because of the nonlinearity of the problem, yield poor values for the orbital elements. Another approach to the problem, due to Gauss, while more complicated, usually produces more accurate results. The reason is that the method of Gauss makes approximations to the dynamics of the motion but treats the geometry of the observations in a precise manner. The error propagation of this approach is generally less unstable than that of the method of Laplace. However, due to the detailed complexity of the method, we will only review the conceptual approach here and refer the student to Danby[7] or Moulton[8] for the details.

Gauss begins by taking advantage of the fact that motion of any object about the sun (or any two body problem) takes place in a plane. Thus it is possible to represent the radius vector from the sun to the object in question for any of the three observations as a linear combination of the other two so that

$$\vec{r}_{pi} = C_j \vec{r}_{pj} + C_k \vec{r}_{pk} \ , \quad i \neq j \neq k; i = 1,2,3 \quad . \tag{7.2.16}$$

These represent three vector equations for the values of $\vec{r}_p$, but they are not linearly independent. However, if we introduce the fact that the observations are made from a moving platform (i.e. the earth) by making use of equation (7.2.5), we can generate three vector equations for the geocentric radius vector of the object $\vec{\rho}_i$ and these are linearly independent. These vector equations are

$$\vec{\rho}_i - C_j \vec{\rho}_j - C_k \vec{\rho}_k = C_j \vec{r}_{\oplus j} + C_k \vec{r}_{\oplus k} - \vec{r}_{\oplus i} \ , \quad i \neq j \neq k; i = 1,2,3 \quad . \tag{7.2.17}$$

If the C_js which determine that fraction of each vector required to produce the third were known then everything on the right-hand side of equations (7.2.17) would be known and we could solve for three values of the geocentric radius vectors $\vec{\rho}_i$. Remember that only the magnitude of $\vec{\rho}_i$ is unknown as the direction cosines are the observations as given in equation (7.2.3). With those three values and the three heliocentric radius vectors of the earth $\vec{r}_{\oplus i}$ we can calculate three values for the heliocentric radius vector of the object $\vec{r}_{pi}$. Given three values for the heliocentric radius vector, there are a number of ways to proceed to obtain the orbital elements. It would appear that there is more information here than is necessary as the three heliocentric radius vectors have nine independent components where only six are required. However, only two of the radius vectors can be regarded as being truly linearly independent. But that is enough. Gauss himself gave a complicated method involving Kepler's equation for obtaining the elements from the three heliocentric radius vectors. Others have used the three heliocentric radius vectors to generate $\dot{\vec{r}}_{p2}$ which, when coupled with $\vec{r}_{p2}$ reduces the problem to the initial value problem that we discussed in Section 7.1. Thus all that remains is to find an expression for the C_js.

Consider taking the vector cross product of equation (7.2.16) with $\vec{r}_{pj}$ to get

$$\vec{r}_{pi} \times \vec{r}_{pj} = r_{pi} r_{pj} \sin\theta_{ij} \hat{\ell} = C_k \vec{r}_{pk} \times \vec{r}_{pj} = C_k r_{pk} r_{pj} \sin\theta_{kj} \hat{\ell} \text{ , } i = 1,2,3 \text{ .} \quad (7.2.18)$$

The vector $\hat{\ell}$ points normal to the orbit and could be used to determine the orbital elements associated with the orientation of the orbit once the $\vec{r}_{pj}$'s are known. The scalar coefficients of $\hat{\ell}$ are the areas of the triangles formed by $\vec{r}_{pi}$ and $\vec{r}_{pj}$ (see Figure 7.3). Thus, the C_ks are given by

$$C_k = \frac{r_{pi} r_{pj} \sin\theta_{ij}}{r_{pk} r_{pj} \sin\theta_{kj}} = \frac{A_{ij}}{A_{kj}} \text{ , } \quad i \neq j \neq k; i = 1,2,3 \quad , \quad (7.2.19)$$

where A_{ij} is the area of the triangle SP_iP_j . If the area of the triangle were the area of the orbital sector enclosed by $\vec{r}_{pi}$ and $\vec{r}_{pj}$ then Kepler's second law would guarantee that C_k would simply be given by the ratio of the appropriate time interval between observations so that

$$C_k \cong \frac{|t_i - t_j|}{|t_k - t_j|} \quad . \quad (7.2.20)$$

Here the linear dependence of the three heliocentric radius vectors is clearly displayed as $C_2 = [C_3/C_l]$ .However, since we only need two heliocentric radii to solve the problem, we may reduce the number of equations in (7.2.17) to just two which will then be linearly independent.

The complicated part of the method of Gauss is involved in calculating corrections to the triangular area so that it will approximate the sector. Since the corrections appear both in A_{ij} and A_{kj} they will tend to cancel to first order and so need not be terribly accurate. This clearly demonstrates the cleverness of Gauss and the reason for the superiority of his method to that of Laplace. The truncation errors of the Taylor series for the time derivatives of ρ enter directly into the determination of the orbital elements. However, the approximation of Gauss is in the geometric representation of the orbital motion of the object and enters only in the second order. Even here, by following the detailed series expansions given by Danby[7] or Moulton[8], one can generally reduce the error in the C_ks and hence the orbital elements to values consistent with the errors of observation. From a rather protracted argument Danby[7] gives the following expressions for the two linearly independent C_is:

$$\left.\begin{array}{l} C_1 = \dfrac{(t_3 - t_2)}{(t_3 - t_1)}\left(1 + \dfrac{k^2}{6r_2^2}\left[(t_3 - t_1)^2 - (t_3 - t_2)^2\right]\right) \\ C_3 = \dfrac{(t_2 - t_1)}{(t_3 - t_1)}\left(1 + \dfrac{k^2}{6r_2^2}\left[(t_3 - t_1)^2 - (t_2 - t_1)^2\right]\right) \end{array}\right\} \quad . \tag{7.2.21}$$

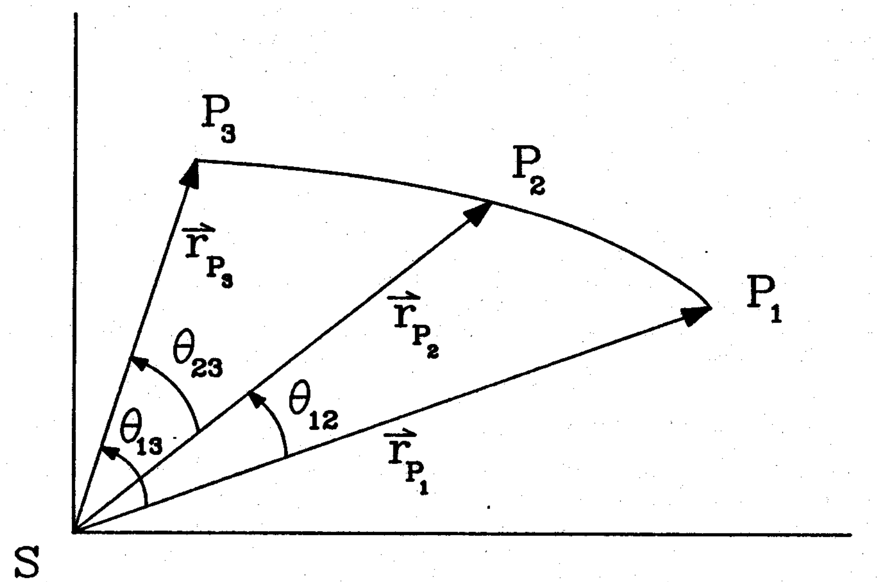

Figure 7.3 shows a section of the orbit of an object revolving about the sun. The object is observed at three points P_i in its orbit and the method of Gauss determines the three heliocentric radius vectors $\vec{r}_{pi}$ .The area A_{ij} is the area of the triangle made from the heliocentric radius vectors $\vec{r}_{pi}$.

The improvements in the estimations of the C_is involve information about the orbit in the form of the factors of $(k^2/6r_2^2)$, as they must, because they involve the corrections required to go from the orbital sectors bounded by the heliocentric radius vectors to the triangles that they form. Danby[7] gives an improved method due to Gibbs which provides a somewhat more accurate form of the approximation, but the concept is the same.

7.3 Degeneracy and Indeterminacy of the Orbital Elements

Before leaving the discussion of orbital elements, I would like to emphasize further that the equations of celestial mechanics are nonlinear. Of the many problems this exacerbates, one is the determination of the orbital elements for the object. Occasionally two of the orbital elements become redundant or indeterminate depending from which end of the two body problem one is starting. For a circular orbit clearly there is no perihelion or point of closest approach. Therefore, there can be no time of perihelion passage. Similarly if the orbital inclination is zero, the orbital plane is co-planar with the plane that defines the coordinate frame and there will be no line of nodes. In this instance the longitude of the ascending node is undefined. One may define the problem out of existence by simply taking the passage of the first point of Aries or the vernal equinox as the reference point for measuring time and the true anomaly. If the inclination is zero and the orbit elliptical, one could simply measure the argument of perihelion from the vernal equinox and have a perfectly well defined orbit, and no trouble would be encountered in locating the object in the sky.

However, in the event that one is determining the orbital elements from observation, there is no advanced information regarding the pathology of the solution. If the inclination is small, the error in the longitude of the ascending node Ω will be large. Similarly, should the eccentricity prove to be very small, the error in the argument of perihelion will be large, so that the time of perihelion passage is poorly known. These errors propagate in a highly nonlinear way and one must be ever mindful of them. The problems caused by a low value of the inclination are not fundamental but result from an unfortunate choice of the coordinate system. They can be eliminated by choosing a different coordinate frame in which to do the calculations. However, the problems are real and will return upon subsequent transformation to the original coordinate frame. The problems introduced by circular orbits are more fundamental as they result from a degeneracy of the orbit itself, and that cannot be transformed away. One can take some comfort from the fact that an uncertain location of the point of perihelion does not mean that the location of the object in its orbit will be uncertain since that error is usually compensated by an opposite error in the time of perihelion passage. The errors in the orbital elements will not be linearly independent so that the net result in locating the object in its orbit will not necessarily be serious. It is better under these conditions to measure the time in the orbit from some well determined location such as the vernal equinox.

In this chapter we have seen how to determine the orbital elements of an object from observational information regarding its motion. This constitutes the second part of the classical celestial mechanics problem of describing the motion of one object about another. In practice, the calculation of precise orbital elements involves many additional practical details concerned with both observations and the theory, but the overall approach is roughly that described here. There are a number of alternative approaches to finding the orbital elements. Indeed, is said that Gauss devised some thirteen different schemes for his doctoral thesis. However, the information content of three sets of angular measurements or the equivalent is always required and the details concern only the devoted practitioner. In the previous chapter we used the elements to predict the motion of the object on the sky. Thus, the two pieces can be put together to predict the motion of an object on the basis of observations of its motion. This is certainly the classical task of any science -that is, to predict the future behavior of the physical world from knowledge of its current behavior. This was a great triumph for Newtonian mechanics in the l7th and l8th centuries and indeed for science itself. The mathematicians and philosophers who came after Newton developed this elegant determinism to deal with much more formidable problems than the two body problem. For the remainder of the book we shall look at some of their successes and some of the remaining problems.

Chapter 7: Exercises

1. Find the altitude, azimuth, Right Ascension, and Declination of the planet Venus as seen from Columbus Ohio at 9:00 PM EST February 10, 1988.

   Given the orbital elements:

   $a = 0.7233316$ AU
   $e = 0.006818 - 0.00005\ T$
   $l = 81^{o}\ 34'\ 19''$ (on Jan 0.5, 1950)
   $i = 3^{o}\ 23'\ 37''.1 + 4''.5 \times T$
   $P = 0.6151856$ yr. = 224.701 days
   $\Omega = 75^{o}\ 47'\ 01'' + 3260'' \times T$
   $\varpi = 130^{o}\ 09'\ 08'' + 5065'' \times T$
   $T = (67 + \text{Date}/365.25)/100$

2. With what geocentric velocity must an artificial satellite be launched horizontally from the earth in order that its apogee distance from the earth's center is 60 earth radii (approximately the moon's distance)? What will be the orbital eccentricity and the orbital period? Ignore air resistance and the gravitational effects of other bodies in the solar system.

3. You plan a trip to Venus. Assume that the orbits of the earth and Venus are circular and co-planar. You will launch your ship from the earth in a direction directly opposite to the earth's orbital motion so that spacecraft has velocity V with respect to the sun when it is "clear of the earth". Note that $V < V_E$ (the earth's orbital speed) and the ship is at its aphelion point at launch. We desire the perihelion point to be at the orbit of Venus ($a = 0.723$AU). What are the semi-major axis (a_r) and the eccentricity (e_r) of the spacecraft's transfer orbit in terms of V? What is the orbital period of the spacecraft? What is the travel time to Venus and where should Venus be in the sky at the time of launch in order to ensure its presence when you arrive?

4. If the heliocentric *Cartesian* coordinates (i.e., the origin is at the sun, the x-axis points toward the vernal equinox, and the z-axis is normal to the ecliptic plane) of a certain comet on November 26.74, 1910 were:

Ᵽ= [(+2.795526), (+1.399919), (0)] AU

and,

o = 267E 16' 36".6
Ω = 206E 40' 11".8
i = 18E 29' 41".1 ,

find the heliocentric coordinates on January 0.5, 1986. Find the altitude and azimuth as seen from Cleveland Ohio at 7:00AM EST. Ignore the difference between ET and UT.

5. Given the heliocentric equatorial position and velocity vectors of an object in orbit about the sun to be

$$\left.\begin{aligned} \vec{r}_0 &= 2\hat{i} + 2\hat{j} + \hat{k} \\ \dot{\vec{r}}_0 &= \frac{\hat{i}}{3} + \frac{\hat{j}}{3} + \hat{k} \end{aligned}\right\} ,$$

where the units of time and distance are years and astronomical units. Find the position of the object two years later. List specifically all assumptions you make and describe clearly the approach you took to the problem.