The Foundations Of Celestial Mechanics

By

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To C.M.Huffer, who taught it the old way,
but who cared that we learn.

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Preface

This book resulted largely from an accident. I was faced with teaching celestial mechanics at The Ohio State University during the Winter Quarter of 1988. As a result of a variety of errors, no textbook would be available to the students until very late in the quarter at the earliest. Since my approach to the subject has generally been non-traditional, a textbook would have been of marginal utility in any event, so I decided to write up what I would be teaching so that the students would have something to review beside lecture notes. This is the result.

Celestial mechanics is a course that is fast disappearing from the curricula of astronomy departments across the country. The pressure to present the new and exciting discoveries of the past quarter century has led to the demise of a number of traditional subjects. In point of fact, very few astronomers are involved in traditional celestial mechanics. Indeed, I doubt if many could determine the orbital elements of a passing comet and predict its future path based on three positional measurements without a good deal of study. This was a classical problem in celestial mechanics at the turn of this century and any astronomer worth his degree would have had little difficulty solving it. Times, as well as disciplines, change and I would be among the first to recommend the deletion from the college curriculum of the traditional course in celestial mechanics such as the one I had twenty five years ago.

There are, however, many aspects of celestial mechanics that are common to other disciplines of science. A knowledge of the mathematics of coordinate transformations will serve well any astronomer, whether observer or theoretician. The classical mechanics of Lagrange and Hamilton will prove useful to anyone who must sometime in a career analyze the dynamical motion of a planet, star, or galaxy. It can also be used to arrive at the equations of motion for objects in the solar system. The fundamental constraints on the N-body problem should be familiar to anyone who would hope to understand the dynamics of stellar systems. And perturbation theory is one of the most widely used tools in theoretical physics. The fact that it is more successful in quantum mechanics than in celestial mechanics speaks more to the relative intrinsic difficulty of the theories than to the methods. Thus celestial mechanics can be used as a vehicle to introduce students to a whole host of subjects that they should know. I feel that this is perhaps the appropriate role for the contemporary study of celestial mechanics at the undergraduate level.

This is not to imply that there are no interesting problems left in celestial mechanics. There still exists no satisfactory explanation for the Kirkwood Gaps of the asteroid belt. The ring system of Saturn is still far from understood. The theory of the motion of the moon may give us clues as to the origin of the moon, but the issue is still far from resolved. Unsolved problems are simply too hard for solutions to be found by any who do not devote a great deal of time and effort to them. An introductory course cannot hope to prepare students adequately to tackle these problems. In addition, many of the traditional approaches to problems were developed to minimize computation by accepting only approximate solutions. These approaches are truly fossils of interest only to those who study the development and history of science. The computational power available to the contemporary scientist enables a more straightforward, though perhaps less elegant, solution to many of the traditional problems of celestial mechanics. A student interested in the contemporary approach to such problems would be well advised to obtain a through grounding in the numerical solution of differential equations before approaching these problems of celestial mechanics.

I have mentioned a number of areas of mathematics and physics that bear on the study of celestial mechanics and suggested that it can provide examples for the application of these techniques to practical problems. I have attempted to supply only an introduction to these subjects. The reader should not be disappointed that these subjects are not covered completely and with full rigor as this was not my intention. Hopefully, his or her appetite will be 'whetted' to learn more as each constitutes a significant course of study in and of itself. I hope that the reader will find some unity in the application of so many diverse fields of study to a single subject, for that is the nature of the study of physical science. In addition, I can only hope that some useful understanding relating to celestial mechanics will also be conveyed. In the unlikely event that some students will be called upon someday to determine the ephemeris of a comet or planet, I can only hope that they will at least know how to proceed.

As is generally the case with any book, many besides the author take part in generating the final product. Let me thank Peter Stoycheff and Jason Weisgerber for their professional rendering of my pathetic drawings and Ryland Truax for reading the manuscript. In addition, Jason Weisgerber carefully proof read the final copy of the manuscript finding numerous errors that evaded my impatient eyes. Special thanks are due Elizabeth Roemer of the Steward Observatory for carefully reading the manuscript and catching a large number of

embarrassing errors and generally improving the result. Those errors that remain are clearly my responsibility and I sincerely hope that they are not too numerous and distracting.

George W. Collins, II
June 24, 1988

Preface to the WEB Edition

It is with some hesitation that I have proceeded to include this book with those I have previously put on the WEB for any who might wish to learn from them. However, recently a past student indicated that she still used this book in the classes she taught and thought it would be helpful to have it available. I was somewhat surprised as the *reason de entra* for the book in the first place was somewhat strained. Even in 1988 few taught celestial mechanics in the manner of the early 20^{th} century before computers made the approach to the subject vastly different. However, the beauty of classical mechanics remains and it was for this that I wrote the book in the first place. The notions of Hamiltonians and Lagrangians are as vibrate and vital today as they were a century ago and anyone who aspires to a career in astronomy or physics should have been exposed to them. There are also similar historical items unique to astronomy to which an aspirant should be exposed. Astronomical coordinate systems and time should be items in any educated astronomer's 'book of knowledge'. While I realize that some of those items are dated, their existence and importance should still be known to the practicing astronomer.

I thought it would be a fairly simple matter to resurrect an old machine readable version and prepare it for the WEB. Sadly, it turned out that all machine-readable versions had disappeared so that it was necessary to scan a copy of the text and edit the result. This I have done in a manner that makes it closely resemble the original edition so as to make the index reasonably useful. The pagination error should be less than ± half a page. The re-editing of the version published by Pachart Publishing House has also afforded me the opportunity to correct a depressingly large number of typographical errors that existed in that effort. However, to think that I have found them all would be pure hubris.

The WEB manuscript was prepared using WORD 2000 and the PDF files generated using ACROBAT 6.0. However, I have found that the ACROBAT 5.0 reader will properly render the files. In order to keep the symbol representation as close to the Pachart Publishing House edition as possible, I have found it necessary to use some fonts that may not be included in the reader's version of WORD. Hence the translation of the PDF's via ACROBAT may suffer. Those fonts are necessary for the correct representation of the Lagrangian in Chapter's 3 and 6 and well as the symbol for the argument of perihelion. The solar symbol use as a subscript may also not be included in the reader's fonts. These fonts are all True Type and in order are:

Commercial Script
WP Greek Helvetica
WP Math A

I believe that the balance of the fonts used is included in most operating systems supporting contemporary word processors. While this may inconvenience some readers, I hope that the reformatting and corrections have made this version more useful.

As with my other efforts, there is no charge for the use of this book, but it is hoped that anyone who finds the book useful would be honest with any attribution that they make.

Finally, I extend my thanks to Professor Andrjez Pacholczyk and Pachart Publishing House for allowing me to release this book on the WEB in spite of the hard copies of the original version that they still have available. Years ago before the internet made communication what it is today, Pacholczyk and Swihart established the Pachart Publishing House partly to make low-volume books such as graduate astronomy text books available to students. I believe this altruistic spirit is still manifest in their decision. I wish that other publishers would follow this example and make some of the out-of-print classics available on the internet.

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April 23, 2004